

GRADIENT POTENTIAL ESTIMATES FOR LINEARIZED MONGE-AMPÈRE EQUATIONS

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ABSTRACT. This paper establishes a pointwise gradient potential estimate for solutions to linearized Monge-Ampère equations and derives a modulus of continuity estimate for the gradient in terms of the associated section adapted potential. As applications, we obtain a sufficient criterion for the continuity of the gradient, local BMO and VMO estimates for the gradient with respect to Monge-Ampère sections, and a local $C^{1,\gamma}$ regularity criterion.

1. INTRODUCTION

In this paper, we investigate local gradient estimates for solutions of the linearized Monge-Ampère equation

$$(1.1) \quad \Phi^{ij} D_{ij} u = f \quad \text{in } \Omega,$$

where $\phi \in C^2(\Omega)$ is strictly convex in a convex domain $\Omega \subset \mathbb{R}^n$,

$$\Phi := (\Phi^{ij}) = \det D^2 \phi (D^2 \phi)^{-1}$$

is the cofactor matrix of $D^2 \phi$ satisfying

$$(1.2) \quad 0 < \lambda \leq \det D^2 \phi \leq \Lambda \quad \text{in } \Omega,$$

for some positive constants λ and Λ .

The linearized Monge-Ampère equation (1.1) plays an important role in many problems. For example, it appears in a class of fourth-order equations of Monge-Ampère type, including the affine mean curvature equation in affine differential geometry [T, TW1, TW2, TW3, TW4] and Abreu's equation arising from complex geometry [D1–D4, Z1, Z2]. In addition, the regularity of the dual semigeostrophic equation for atmospheric modeling is also related to the linearized Monge-Ampère equation [Lo, Le2]. A new application has recently been found in the study of convex functionals with a convexity constraint arising in the Rochet-Choné model for the monopolist problem in economics [Le4, LZ, KLWZ], which involves singular Abreu's equations.

Date: June 27, 2026.

2020 Mathematics Subject Classification. 35B65, 35J96, 31C45.

Key words and phrases. Gradient potential estimates, borderline-type regularity, linearized Monge-Ampère equations.

This research is partially supported by 2023YFA009900 and NSFC Grant 12271008.

Under assumption (1.2), the linearized Monge-Ampère equation (1.1) fails to be uniformly elliptic in general, making the analysis challenging. The first breakthrough in the regularity theory for the equation (1.1) was made by Caffarelli and Gutiérrez. In [CG], they established the Harnack inequality for the homogeneous linearized Monge-Ampère equation, which yields the interior Hölder continuity of solutions. Since then, many further developments have been obtained. Hölder estimates for the non-homogeneous equation have been extensively studied in [Lo, LS, LN, Le1, Le2, Le3, Wa, Ki, CWZ, GL]. In particular, when the right-hand side is a signed Radon measure, a potential estimate and interior Hölder estimates were obtained recently in [CWZ, GL].

Concerning gradient regularity, assuming (1.2) together with the continuity of $\det D^2\phi$, Gutiérrez and Nguyen [GN1] established the Cordes-Nirenberg type interior $C^{1,\alpha}$ estimates for solutions of the linearized Monge-Ampère equation with $f \in L^q(\Omega)$, $q > n$. When $f \in L^q(\Omega)$ with $n/2 < q \leq n$, the interior and global $W^{1,p}$ estimates were established in [LN], namely, $u \in W^{1,p}(\Omega)$ for all $p < \frac{nq}{n-q}$. In the borderline case, Du may fail to be continuous. Indeed, even in the model case $\phi = \frac{1}{2}|x|^2$, namely for the Poisson equation, Du may be unbounded [Ci]. In this paper, we establish the gradient potential estimate.

Recall that for the Poisson equation $\Delta u = f$, we have the well-known gradient potential estimate

$$|Du(x)| \leq C\mathbf{I}_1^f(x, r) + C \int_{B(x, r)} |Du| \, dy$$

for a.e. x , where $\mathbf{I}_1^f(x, r)$ is the truncated Riesz potential of f defined by

$$\mathbf{I}_1^f(x, r) = \int_0^r \frac{|\mu|(B(x, \rho))}{\rho^{n-1}} \frac{d\rho}{\rho} = \int_0^r \int_{B(x, \rho)} |f(y)| \, dy \, d\rho,$$

with $\mu := f \, dx$ and

$$\int_{B(x, r)} |Du| \, dy := \frac{1}{|B(x, r)|} \int_{B(x, r)} |Du| \, dy.$$

For fully nonlinear uniform elliptic equations

$$F(x, D^2u) = f,$$

the following gradient potential estimate was established in [DKM]:

$$|Du(x)| \leq C\mathbf{I}_p^f(x, r) + C \left(\int_{B(x, r)} |Du|^q \, dx \right)^{1/q}$$

for a.e. x , all $p > n_E$ and $q > n$, where n_E is a constant depending only on n and the ellipticity of F , and $\mathbf{I}_p^f(x, r)$ is the L^p version of the classical Riesz potential defined by

$$\mathbf{I}_p^f(x, r) := \int_0^r \left(\int_{B(x, \rho)} |f(y)|^p \, dy \right)^{1/p} d\rho \geq \mathbf{I}_1^f(x, r).$$

Because of the hidden nonlinear structure, degeneracy and affine invariance of the linearized Monge-Ampère equation, we use the section-adapted modified Riesz potential (the L^q version of the classical Riesz potential) defined by

$$(1.3) \quad \mathbf{I}_{\phi,q}^{f,\alpha}(x_0, h) := h^{\alpha-\frac{1}{2}} \int_0^h \left(\int_{S_\phi(x_0, \rho)} |f(y)|^q dy \right)^{1/q} \frac{d\rho}{\rho^\alpha},$$

for any $q \geq 1$, $\alpha \geq \frac{1}{2}$, where

$$S_\phi(x_0, h) := \{x \in \Omega : \phi(x) < \phi(x_0) + D\phi(x_0) \cdot (x - x_0) + h\}$$

is a section of $\phi \in C^2(\Omega)$ centered at $x_0 \in \Omega$ with height $h > 0$. Under the assumption (1.2), sections are equivalent to Euclidean balls in terms of their geometric and measure-theoretic properties; see [Le5] for an overview of these properties.

For convenience, we denote

$$\det D^2\phi := g \in C(\Omega).$$

We do not address the question of optimal regularity assumptions on g . Throughout the paper, we assume that ϕ is smooth. Nevertheless, all estimates depend only on the structural constants appearing in the assumptions and are independent of this auxiliary smoothness assumption. We have the following pointwise gradient potential estimate:

Theorem 1.1 (Gradient potential estimate). *Let $\phi \in C^2(\Omega)$ be a strictly convex function on Ω satisfying*

$$(1.4) \quad 0 < \lambda \leq g \leq \Lambda \text{ in } \Omega, \quad \text{and} \quad g \in C(\Omega).$$

Let $f \in L^q(\Omega)$ with $\frac{n}{2} < q \leq n$, and let u solve (1.1). Choose p so that

$$n < p < \frac{nq}{n-q} \quad \text{if } q < n, \quad \text{and} \quad n < p < \infty \quad \text{if } q = n.$$

For any $\theta \in (0, 1)$ and $\alpha \in (1/2, 1)$, for any Lebesgue point $x_0 \in \Omega'$ of Du with $\Omega' \Subset \Omega$, there exist $C, h_ > 0$, depending only on $n, \alpha, \theta, \lambda, \Lambda, p, q, \text{dist}(\Omega', \partial\Omega)$ and the modulus of continuity of g , such that whenever $0 < h \leq h_*$ and $S_\phi(x_0, ch^\theta) \Subset \Omega$, one has*

$$(1.5) \quad |Du(x_0)| \leq C \int_{S_\phi(x_0, h)} |Du(x)| dx + C \left(\int_{S_\phi(x_0, h^\theta)} |Du(x)|^p dx \right)^{1/p} + C \mathbf{I}_{\phi,q}^{f,\alpha}(x_0, ch^\theta),$$

where $c > 0$ is a constant depending only on $n, \theta, \lambda, \Lambda, \text{dist}(\Omega', \partial\Omega)$ and the modulus of continuity of g .

Remark 1.2. *We give several remarks on the modified Riesz potential (1.3):*

- (1) When $\phi = \frac{1}{2}|x|^2$ and $\alpha = \frac{1}{2}$, we have

$$\mathbf{I}_{\phi,q}^{f,\alpha}(x_0, h) = \sqrt{2}\mathbf{I}_q^f(x_0, \sqrt{2h}),$$

where $\mathbf{I}_q^f(x_0, h)$ is the usual L^q -version Riesz potential defined in [DKM].

- (2) We can also define the section-adapted Wolff potential as

$$(1.6) \quad W_{\phi,\beta,q}^\mu(x, h) := \int_0^h \left(\frac{|\mu|(S_\phi(x, \rho))}{\rho^{\frac{1}{2}(n-\beta q)}} \right)^{\frac{1}{q-1}} \frac{d\rho}{\rho},$$

for $0 < \beta \leq n/q$. When $\phi = \frac{1}{2}|x|^2$, we can derive

$$W_{\phi,\beta,q}^\mu(x, h) = C_{n,\beta,q} W_{\beta,q}^\mu(x, \sqrt{2h}),$$

where $W_{\beta,q}^\mu(x, r)$ is the usual Wolff potential. Then we can write (1.3) as

$$(1.7) \quad I_{\phi,q}^{f,\alpha}(x_0, h) \approx h^{\alpha-\frac{1}{2}} W_{\phi,\beta,q+1}^{|f|^q}(x_0, h) \quad \beta := \frac{2(1-\alpha)q}{q+1}.$$

Remark 1.3. We give several remarks on the gradient potential estimate (1.5):

- (1) The modified Riesz potential $\mathbf{I}_{\phi,q}^{f,\alpha}(x_0, h)$ can be seen as an integrated version of the quantity $N_{\phi,f,q}(x_0)$ defined in [LN] by

$$N_{\phi,f,q}(x_0) := \sup_{h \leq h_0} h^{\frac{1-\varepsilon}{2}} \left(\int_{S_\phi(x_0, h)} |f(x)|^q dx \right)^{1/q}$$

for $\varepsilon \in (0, 1)$ and $h_0 > 0$. Let $\alpha \in (\frac{1}{2}, \frac{1+\varepsilon}{2})$, we derive

$$\begin{aligned} \mathbf{I}_{\phi,q}^{f,\alpha}(x_0, h_0) &= h_0^{\alpha-\frac{1}{2}} \int_0^{h_0} \left(\int_{S_\phi(x_0, \rho)} |f(x)|^q dx \right)^{1/q} \frac{d\rho}{\rho^\alpha} \\ &\leq \left(\sup_{\rho \leq h_0} \rho^{\frac{1-\varepsilon}{2}} \left(\int_{S_\phi(x_0, \rho)} |f(x)|^q dx \right)^{1/q} \right) h_0^{\alpha-\frac{1}{2}} \int_0^{h_0} \rho^{-\frac{1-\varepsilon}{2}} \frac{d\rho}{\rho^\alpha} \\ &\leq C_{\varepsilon,\alpha} h_0^{\varepsilon/2} N_{\phi,f,q}(x_0). \end{aligned}$$

Consequently, after estimating the local averages of Du in Theorem 1.1 by the standard local $W^{1,p}$ estimate, one recovers a pointwise gradient bound of the same form as that in [LN], namely

$$|Du(x_0)| \leq C(\|u\|_{L^\infty(\Omega)} + N_{\phi,f,q}(x_0)).$$

Therefore Theorem 1.1 should be viewed as complementary to the pointwise estimate of [LN] in a different form.

- (2) The gradient potential estimate (1.5) can yield directly the following gradient estimate in Lorentz spaces (see Definition 2.5):

$$(1.8) \quad \|Du\|_{L(r,\gamma)(\Omega')} \leq C(\|u\|_{L^\infty(\Omega)} + \|f\|_{L(s,\gamma)(\Omega)}),$$

for any $s \in (n/2, n]$, $r \in (n, ns/(n-s))$ and $\gamma \in (0, \infty]$. The gradient estimate (1.8) can be viewed as an analogue of the classical $W^{1,p}$ estimates of [LN] in Lorentz spaces. One way to justify (1.8) is to combine (1.5), (1.7), the Havin-Mazya type inequality

$$W_{\phi,\beta,q}^\mu(x, h) \leq C \mathbf{I}_{\phi,\beta} \left[(\mathbf{I}_{\phi,\beta}(|\mu|))^{\frac{1}{q-1}} \right] (x, Ch),$$

and the Lorentz estimate for $\mathbf{I}_{\phi,\beta}$

$$\|\mathbf{I}_{\phi,\beta}(f)(\cdot, h_0)\|_{L(\frac{ns}{n-\beta s}, \gamma)(\Omega')} \leq C \|f\|_{L(s, \gamma)(\Omega)}.$$

Equivalently, one may obtain (1.8) more directly from the $W^{1,p}$ -estimate of [LN] together with standard Lorentz space properties. Therefore, (1.8) is recorded here only as a consequence and not as the main new contribution.

The potential estimate also yields a modulus of continuity for Du :

Theorem 1.4 (Gradient continuity via potential). *Under the assumptions of Theorem 1.1, assume that for every $\Omega' \Subset \Omega$,*

$$(1.9) \quad \lim_{h \rightarrow 0} \sup_{\substack{x \in \Omega', \\ S_\phi(x, h) \Subset \Omega}} \mathbf{I}_{\phi,q}^{f,\alpha}(x, h) = 0.$$

Then $u \in C_{loc}^1(\Omega)$. Moreover, for every $\Omega' \Subset \Omega'' \Subset \Omega$ and every $\theta \in (0, 1)$, we have

$$(1.10) \quad |Du(x_1) - Du(x_2)| \leq C \|Du\|_{L^\infty(\Omega'')} |x_1 - x_2|^{\kappa\beta} + C \max_{x \in \{x_1, x_2\}} \mathbf{I}_{\phi,q}^{f,\alpha}(x, c|x_1 - x_2|^{\theta\beta})$$

for all $x_1, x_2 \in \Omega'$, where $C > 0$ depends only on $n, \alpha, p, q, \theta, \lambda, \Lambda, \Omega', \Omega''$ and the modulus of continuity of g , $c > 0$ depends only on $n, p, q, \theta, \lambda, \Lambda, \Omega'$ and the modulus of continuity of g , $\kappa > 0$ depends only on n, p, q, α and θ , and $\beta > 1$ depends only on $n, \lambda, \Lambda, p, q$ and Ω' . Here β comes from the local $C^{1,\beta-1}$ regularity of the Monge-Ampère potential ϕ .

We next establish several borderline-type regularity results as consequences of the potential estimate. These should be regarded as sufficient conditions formulated in the language of section geometry. We make no claim regarding their optimality. Related borderline regularity theories have been developed for p -Laplacian type equations [DM1, DM2, DZ, Mi] and fully nonlinear equations [DKM].

Corollary 1.5. *Let $\phi \in C^2(\Omega)$ be a strictly convex function on Ω satisfying (1.4) and u be a solution of (1.1). Fix $q \in (n/2, n]$ and $\alpha \in (1/2, 1)$. For $\Omega' \Subset \Omega$, set*

$$\omega_{\alpha,q}(r, \Omega') := \sup_{\substack{x \in \Omega', 0 < h < r \\ S_\phi(x, h) \Subset \Omega}} \left(h^{q(1-\alpha) - \frac{n}{2}} \int_{S_\phi(x, h)} |f(y)|^q dy \right)^{1/q}.$$

Then

- (1) if $\omega_{\alpha,q}(r, \Omega') \rightarrow 0$ as $r \rightarrow 0$ for every $\Omega' \Subset \Omega$, then $Du \in VMO_{loc}(\Omega, \phi)$ (see Definition 2.7);

(2) if

$$\sup_{0 < r \leq r_0} \omega_{\alpha, q}(r, \Omega') < \infty$$

for every $\Omega' \Subset \Omega$ and some $r_0 > 0$, then $Du \in BMO_{loc}(\Omega, \phi)$ (see Definition 2.7);

(3) if for some $\varepsilon \in (0, 1)$,

$$\sup_{S_\phi(x, h) \Subset \Omega} h^{\frac{1-\varepsilon}{2}} \left(\int_{S_\phi(x, h)} |f(y)|^n dy \right)^{1/n} < \infty,$$

then $u \in C_{loc}^{1, \gamma}(\Omega)$ for some $\gamma > 0$.

In particular,

- (1) if $f \in L(s, \infty)$ locally in Ω for some $s > n$, then $Du \in BMO_{loc}(\Omega, \phi)$;
- (2) if $f \in L^r(\Omega)$ for some $r > n$, then $u \in C_{loc}^{1, \gamma}(\Omega)$ for some $\gamma > 0$.

Remark 1.6. The conditions in Corollary 1.5 are sufficient conditions expressed in terms of sections. They are not asserted to be sharp. The endpoint cases, such as the endpoint in the Lorentz estimate, require additional arguments and are left open in Question 1.7.

Question 1.7. Compared with uniformly elliptic equations, the present approach loses a small amount of scale because of the degeneracy and affine normalization of (1.1). We therefore ask whether Theorem 1.1 remains valid when $\theta = 1$, and whether the endpoint $r = \frac{ns}{n-s}$ in (1.8) can be obtained.

The paper is organized as follows. In Section 2, we collect some useful lemmas that will be used later. We prove the gradient potential estimate (Theorem 1.1) in Section 3 and derive gradient-continuity estimates (Theorem 1.4) and borderline-type regularity results (Corollary 1.5) in Section 4.

2. PRELIMINARIES

2.1. Geometry of sections. For a strictly convex function ϕ defined in Ω and $h > 0$, we denote by $S_h(\phi)$ the section of ϕ centered at its minimum point y_0 with height h , i.e.

$$S_h(\phi) := \left\{ x \in \Omega : \phi(x) \leq \min_{\Omega} \phi + h \right\} = S_\phi(y_0, h).$$

Lemma 2.1. [GN1, Lemma 3.2] Suppose $B_1(0) \subset \Omega \subset B_n(0)$ is a normalized convex domain. Then there exist universal constants $h_0, \tau_0 > 0$ and a positive definite matrix $M = A^t A$ satisfying

$$\det M = 1, \quad \text{and} \quad 0 < c_1 I \leq M \leq c_2 I,$$

such that if $\phi \in C^2(\Omega)$ is a strictly convex function in Ω with

$$1 - \varepsilon \leq \det D^2 \phi \leq 1 + \varepsilon \text{ in } \Omega, \quad \text{and} \quad \phi = 0 \text{ on } \partial\Omega,$$

then for $0 < h \leq h_0$ and $\varepsilon \leq \tau_0 h^2$, we have

$$B_{(1-C(h^{1/2}+h^{-1}\varepsilon^{1/2}))\sqrt{2}}(0) \subset h^{-1/2}TS_h(\phi) \subset B_{(1+C(h^{1/2}+h^{-1}\varepsilon^{1/2}))\sqrt{2}}(0),$$

where $Tx := A(x - x_0)$ and $x_0 \in \Omega$ is the minimum point of ϕ .

Lemma 2.2. [GN1, Lemma 3.3] Suppose Ω is a convex domain satisfying

$$B_{(1-\sigma)\sqrt{2}}(0) \subset \Omega \subset B_{(1+\sigma)\sqrt{2}}(0)$$

with $0 < \sigma \leq \frac{1}{4}$. There exist $\mu_0, \tau_0 > 0$ independent of σ , and a positive definite matrix $M = A^t A$, with

$$\det M = 1 \quad \text{and} \quad (1 - C\sigma)I \leq M \leq (1 + C\sigma)I,$$

such that if $\phi \in C^2(\Omega)$ is a strictly convex function in Ω satisfying

$$1 - \varepsilon \leq \det D^2\phi \leq 1 + \varepsilon \quad \text{in } \Omega, \quad \text{and} \quad \phi = 0 \quad \text{on } \partial\Omega,$$

then for $\mu \in (0, \mu_0)$ and $\varepsilon \leq \tau_0 \mu^2$ we have

$$B_{(1-C(\sigma\mu^{1/2}+\mu^{-1}\varepsilon^{1/2}))\sqrt{2}}(0) \subset \mu^{-1/2}TS_\mu(\phi) \subset B_{(1+C(\sigma\mu^{1/2}+\mu^{-1}\varepsilon^{1/2}))\sqrt{2}}(0),$$

where $Tx := A(x - x_0)$ and $x_0 \in \Omega$ is the minimum point of ϕ .

Lemma 2.3. Let $\phi \in C^2(\Omega)$ be a strictly convex function in Ω . For any $f \in L^\infty(\Omega)$ and any fixed $h > 0$, we define

$$F(x) = (f)_{S_\phi(x,h)} := \frac{1}{|S_\phi(x,h)|} \int_{S_\phi(x,h)} f(y) \, dy$$

whenever $S_\phi(x,h) \Subset \Omega$. Then F is continuous.

Proof. Let $x_0 \in \Omega$ and choose $h > 0$ such that $S_\phi(x_0, 2h) \Subset \Omega$. Set $M := \|f\|_{L^\infty(\Omega)}$. If $M = 0$, then $F \equiv 0$, and the conclusion is trivial. Hence we assume $M > 0$. We define the function

$$\Psi(x, y) := \phi(y) - \phi(x) - D\phi(x) \cdot (y - x).$$

Then

$$S_\phi(x, h) = \{y \in \Omega : \Psi(x, y) < h\},$$

and the function Ψ is continuous in (x, y) .

We first prove that

$$|S_\phi(x, h) \Delta S_\phi(x_0, h)| \rightarrow 0 \quad \text{as } x \rightarrow x_0.$$

Let $0 < t < h$. We claim that there exists $\delta_t > 0$ such that if $|x - x_0| < \delta_t$, then

$$S_\phi(x_0, h - t) \subset S_\phi(x, h) \subset S_\phi(x_0, h + t).$$

Indeed, since Ψ is continuous, we may choose $\delta_t > 0$ small enough such that

$$|\Psi(x, y) - \Psi(x_0, y)| < t$$

for all $y \in \overline{S_\phi(x_0, h - t)}$ whenever $|x - x_0| < \delta_t$. Thus, if $y \in S_\phi(x_0, h - t)$, then

$$\Psi(x, y) \leq \Psi(x_0, y) + t < h.$$

Hence $y \in S_\phi(x, h)$, and therefore $S_\phi(x_0, h - t) \subset S_\phi(x, h)$.

It remains to prove the second inclusion. Let $U := S_\phi(x_0, h + t)$. For every $z \in \partial U$, we have $\Psi(x_0, z) = h + t$. Using the continuity of Ψ and the compactness of ∂U , and taking $\delta_t > 0$ smaller if necessary, we may assume that

$$\Psi(x, z) > h \quad \text{for all } z \in \partial U$$

whenever $|x - x_0| < \delta_t$. We may also assume that $x \in U$, since $x_0 \in U$ and U is open. Suppose, by contradiction, that there exists

$$y \in S_\phi(x, h) \setminus S_\phi(x_0, h + t).$$

Since $x \in U$, there exists $z_0 \in [x, y] \cap \partial U$. The convexity of Ψ in y yields $\Psi(x, z_0) < h$, which contradicts the assumption. Therefore $S_\phi(x, h) \subset S_\phi(x_0, h + t)$.

Consequently, for $|x - x_0| < \delta_t$, we have

$$S_\phi(x, h) \triangle S_\phi(x_0, h) \subset S_\phi(x_0, h + t) \setminus S_\phi(x_0, h - t).$$

The convexity of sections yields $|\partial S_\phi(x_0, h)| = 0$. Therefore

$$|S_\phi(x_0, h + t) \setminus S_\phi(x_0, h - t)| \rightarrow 0.$$

It follows that

$$|S_\phi(x, h) \triangle S_\phi(x_0, h)| \rightarrow 0 \quad \text{as } x \rightarrow x_0.$$

In particular,

$$||S_\phi(x, h)| - |S_\phi(x_0, h)|| \leq |S_\phi(x, h) \triangle S_\phi(x_0, h)| \rightarrow 0.$$

Thus, for x sufficiently close to x_0 , we have

$$|S_\phi(x, h)| \geq \frac{1}{2} |S_\phi(x_0, h)|.$$

Now set

$$S_x := S_\phi(x, h), \quad S_0 := S_\phi(x_0, h).$$

Hence, for x sufficiently close to x_0 , we have

$$\begin{aligned} |F(x) - F(x_0)| &= \left| \frac{1}{|S_x|} \int_{S_x} f(y) \, dy - \frac{1}{|S_0|} \int_{S_0} f(y) \, dy \right| \\ &\leq \frac{1}{|S_x|} \left| \int_{S_x} f(y) \, dy - \int_{S_0} f(y) \, dy \right| + \left| \frac{1}{|S_x|} - \frac{1}{|S_0|} \right| \int_{S_0} |f(y)| \, dy \\ &\leq \frac{2M}{|S_0|} |S_x \triangle S_0| + \frac{2M}{|S_0|} |S_x \triangle S_0| = \frac{4M}{|S_0|} |S_x \triangle S_0| \rightarrow 0, \end{aligned}$$

as $x \rightarrow x_0$. Therefore, F is continuous. $\square$

Remark 2.4. *The continuity property in Lemma 2.3 does not essentially depend on the smoothness of ϕ . If ϕ is merely convex, we may use a chosen subgradient. Namely, for $p \in \partial\phi(x)$, set*

$$S_\phi(x, p, h) := \{y \in \Omega : \phi(y) < \phi(x) + p \cdot (y - x) + h\}.$$

If

$$x_k \rightarrow x_0, \quad p_k \in \partial\phi(x_k), \quad p_k \rightarrow p_0 \in \partial\phi(x_0), \quad \text{and} \quad S_\phi(x_0, p_0, 2h) \Subset \Omega,$$

then

$$|S_\phi(x_k, p_k, h) \triangle S_\phi(x_0, p_0, h)| \rightarrow 0.$$

Consequently, for every $f \in L^\infty(\Omega)$,

$$\frac{1}{|S_\phi(x_k, p_k, h)|} \int_{S_\phi(x_k, p_k, h)} f(y) \, dy \rightarrow \frac{1}{|S_\phi(x_0, p_0, h)|} \int_{S_\phi(x_0, p_0, h)} f(y) \, dy.$$

Thus, without differentiability, the natural continuity statement holds with respect to the pair (x, p) , where $p \in \partial\phi(x)$, rather than x alone.

2.2. Lorentz spaces.

Definition 2.5. For any $0 < p, q \leq \infty$, we define

$$\|f\|_{L^{p,q}(\Omega)} := \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{1/q}, & q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), & q = \infty, \end{cases}$$

where

$$f^*(t) := \inf\{s > 0 : |\{x \in \Omega : |f(x)| > s\}| \leq t\}.$$

The Lorentz spaces are defined by

$$L^{p,q}(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \text{ measurable} : \|f\|_{L^{p,q}(\Omega)} < \infty\}.$$

Lemma 2.6. (Hölder-type inequality) For $1 \leq p, p_1, p_2 < \infty$ and $1 < q, q_1, q_2 \leq \infty$ satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q},$$

there exists $C_{p_1, p_2, q_1, q_2} > 0$ such that

$$(2.1) \quad \|fg\|_{L^{p,q}(\Omega)} \leq C_{p_1, p_2, q_1, q_2} \|f\|_{L^{p_1, q_1}(\Omega)} \cdot \|g\|_{L^{p_2, q_2}(\Omega)}.$$

Definition 2.7. [H] For $f \in L^1(\Omega)$ and $A \subset \Omega$, the mean oscillation of f over A is defined by

$$\text{mosc}_A f := \int_A |f(x) - (f)_A| \, dx, \quad \text{where } (f)_A := \int_A f(y) \, dy.$$

Let $\phi \in C^2(\Omega)$ be a strictly convex function on Ω satisfying (1.2). For any $\Omega' \Subset \Omega$, we write

$$Q_f(r, \Omega') := \sup_{\substack{x_0 \in \Omega' \\ \text{diam}(S_\phi(x_0, h)) \leq r}} \text{mosc}_{S_\phi(x_0, h)} f.$$

We say $f \in BMO_{loc}(\Omega, \phi)$ and $f \in VMO_{loc}(\Omega, \phi)$ if, respectively,

$$\limsup_{r \rightarrow 0} Q_f(r, \Omega') < \infty \quad \text{and} \quad \lim_{r \rightarrow 0} Q_f(r, \Omega') = 0$$

for any $\Omega' \Subset \Omega$.

2.3. Interior $W^{1,p}$ estimates.

Lemma 2.8. *Let $\phi \in C^2(\Omega)$ be a convex function in a normalized domain $\Omega \subset \mathbb{R}^n$ with $S_1 := S_\phi(0, 1) \Subset \Omega$ and $f \in L^q(\Omega)$ with $\frac{n}{2} < q \leq n$. Choose p so that*

$$n < p < \frac{nq}{n-q} \quad \text{if } q < n, \quad \text{and} \quad n < p < \infty \quad \text{if } q = n.$$

Suppose u is a solution of (1.1) in Ω . Then there exists $\delta \in (0, 1)$ depending only on n, p and q such that if

$$\begin{cases} 1 - \delta \leq \det D^2\phi \leq 1 + \delta, & \text{in } S_1, \\ \phi = 0 & \text{on } \partial S_1, \end{cases}$$

then

$$\left(\int_{S_{1/2}} |Du|^p dx \right)^{1/p} \leq C \|u\|_{L^\infty(S_1)} + C \left(\int_{S_1} |f|^q dx \right)^{1/q},$$

where $C = C(n, p, q)$.

Proof. The proof is similar to [LN, Theorem 1.3], which is a direct combination of [LN, Theorem 1.2] and [GN2, Theorem 2.2]. $\square$

2.4. $C^{1,1}$ estimates for uniformly elliptic equations. The following form of the standard interior estimate is used only on subdomains separated from the boundary. It follows from Pogorelov-type estimates for the solution w of (2.2) and the classical interior $C^{1,1}$ theory for the resulting uniformly elliptic linear equation; see [LN, Lemma 3.3] and [Gu].

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^n$ be a normalized convex domain with $B_1(0) \subset \Omega \subset B_n(0)$ and let $w \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be a convex solution of*

$$(2.2) \quad \begin{cases} \det D^2w = 1 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

- (i) *Let $x_1 \in \Omega$ be the minimum point of w . Then $|w(x_1)| \sim c_n$ for some universal constant c_n and we have the Pogorelov's estimates:*

$$\frac{2}{C_2^2} I \leq D^2w \leq \frac{2}{C_1^2} I \quad \text{for all } x \in \Omega \text{ with } \text{dist}(x, \partial\Omega) \geq c_n,$$

where C_1 and C_2 are constants depending only on n .

- (ii) *For any solution $v \in C^2(B_1)$ of $W^{ij} D_{ij}v = 0$ in $B_1(0)$, where $W = (W^{ij}) := \det D^2w (D^2w)^{-1}$, we have the classical interior $C^{1,1}$ estimate*

$$\|v\|_{C^{1,1}(B_{1/2}(0))} \leq c \|v\|_{L^\infty(\partial B_{3/4}(0))}$$

where c is a constant depending only on n .

2.5. An approximation lemma. The following approximation is a direct consequence of [GN1, LN].

Lemma 2.10. *Let $\phi \in C^2(\Omega)$ be a strictly convex function in a convex domain $\Omega \subset \mathbb{R}^n$. Assume that $S_1 := S_\phi(0, 1) \Subset \Omega$, S_1 is normalized, i.e. $B_1(0) \subset S_1 \subset B_n(0)$, and 0 is the minimum point of ϕ . Let $f \in L^q(\Omega)$ with $\frac{n}{2} < q \leq n$ and u be a solution of (1.1) with $\|u\|_{L^\infty(S_1)} \leq M$ for some constant M . Then, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, n, q, M) > 0$ such that if*

$$\begin{cases} 1 - \delta \leq \det D^2\phi \leq 1 + \delta & \text{in } S_1, \\ \phi = 0 & \text{on } \partial S_1, \end{cases} \quad \text{and} \quad \left(\int_{S_1} |f(x)|^q dx \right)^{1/q} \leq \delta,$$

then the solution v of

$$\begin{cases} W^{ij} D_{ij} v = 0 & \text{in } S_{1/2}, \\ v = u & \text{on } \partial S_{1/2}, \end{cases} \quad \text{where} \quad \begin{cases} \det D^2 w = 1 & \text{in } S_1, \\ w = 0 & \text{on } \partial S_1, \end{cases}$$

satisfies

$$\|u - v\|_{L^\infty(S_{1/4})} < \varepsilon,$$

where W is the cofactor matrix of $D^2 w$.

Proof. This is an extension of [GN1, Lemma 4.2] from $q = n$ to $q \in (n/2, n]$. One can use [LN, Lemma 2.5] and [LN, Lemma 2.6] to obtain the estimates on $\|u - v\|_{L^\infty(S_{1/4})}$ by taking δ sufficiently small. $\square$

3. THE GRADIENT POTENTIAL ESTIMATES

For the rest of the paper, we consider the excess functional

$$E_p(S) := \left(\int_S |Du - (Du)_S|^p dx \right)^{1/p},$$

where

$$(Du)_S := \int_S Du(x) dx,$$

and $S \subset \Omega$ denotes a section with positive height and $p \geq 1$, and u is a solution of (1.1). We have the following crucial decay estimate:

Lemma 3.1. *Let $\phi \in C^2(\Omega)$ be a strictly convex function in a normalized convex domain $\Omega \subset \mathbb{R}^n$ with $S_1 := S_\phi(0, 1) \Subset \Omega$, where 0 is the minimum point of ϕ , and $f \in L^q(\Omega)$ with $\frac{n}{2} < q \leq n$. Let u be a solution of (1.1). Given p with $n < p < \frac{nq}{n-q}$ if $q < n$, and $n < p < \infty$ if $q = n$, and $c_0 \in (0, 1)$, there exist $\delta, \sigma \in (0, 1)$, both depending only on n, p, q and c_0 such that if*

$$E_p(S_1) \leq 1, \quad \left(\int_{S_1} |f(x)|^q dx \right)^{1/q} \leq \delta$$

and

$$\begin{cases} 1 - \delta \leq \det D^2\phi \leq 1 + \delta, & \text{in } S_1, \\ \phi = 0 & \text{on } \partial S_1, \end{cases}$$

then

$$E_p(S_\sigma) \leq c_0,$$

where $S_\sigma := S_\phi(0, \sigma)$.

Proof. We assume that $(u)_{S_1} = 0$ and $(Du)_{S_1} = 0$. Otherwise, we consider

$$\tilde{u} = u - (u)_{S_1} - (Du)_{S_1} \cdot x,$$

which solves the same equation as u . Since $p > n$, Morrey's inequality gives

$$\|u\|_{L^\infty(S_1)} \leq C \left(\int_{S_1} |Du|^p dx \right)^{1/p} = CE_p(S_1) \leq C =: M.$$

Let w be the solution of

$$\begin{cases} \det D^2w = 1 & \text{in } S_1, \\ w = 0 & \text{on } \partial S_1. \end{cases}$$

Then Lemma 2.10 implies that for any $\varepsilon \in (0, M)$, there exist $\delta_1 = \delta_1(\varepsilon, n, q) \in (0, 1)$ and v solving

$$\begin{cases} W^{ij} D_{ij}v = 0 & \text{in } S_{1/2}, \\ v = u & \text{on } \partial S_{1/2}, \end{cases}$$

such that if

$$\begin{cases} 1 - \delta_1 \leq \det D^2\phi \leq 1 + \delta_1 & \text{in } S_1, \\ \phi = 0 & \text{on } \partial S_1, \end{cases}$$

and

$$\left(\int_{S_1} |f(x)|^q \right)^{1/q} \leq \delta_1,$$

then

$$\|u - v\|_{L^\infty(S_{1/4})} < \varepsilon.$$

Then

$$\|v\|_{L^\infty(S_{1/4})} \leq \|u\|_{L^\infty(S_1)} + \|u - v\|_{L^\infty(S_{1/4})} \leq M + \varepsilon \leq 2M.$$

Applying the $C^{1,1}$ -regularity estimate in Lemma 2.9, we have

$$\|v\|_{C^{1,1}(S_{1/8})} \leq C\|v\|_{L^\infty(S_{1/4})} \leq CM.$$

It follows that there exists an affine function $l = v(0) + Dv(0) \cdot x$ such that

$$\|v - l\|_{L^\infty(B_r(0))} \leq CMr^2$$

for all small $r \leq r_0$ such that $B_{r_0}(0) \subset S_{1/8}$. By Lemma 2.1 we choose

$$\delta_2 = \delta_2(\sigma) \leq \tau_0 \sigma^2$$

such that $S_{2\sigma} \subset B_{c\sqrt{\sigma}}$ for all $\sigma \leq \sigma_0$. Then for all $\sigma \leq \sigma_0$ with $c\sqrt{\sigma_0} = r_0$, we have

$$\|v - l\|_{L^\infty(S_{2\sigma})} \leq \|v - l\|_{L^\infty(B_{c\sqrt{\sigma}})} \leq CM\sigma.$$

We claim that there exists $\delta_3 \leq \tau_0(2\sigma)^2$ such that if

$$|\det D^2\phi - 1| < \delta_3 \text{ in } S_1$$

and $\phi = 0$ on ∂S_1 , then

$$(3.1) \quad \left(\int_{S_\sigma} |Du|^p dx \right)^{1/p} \leq \frac{C}{\sqrt{\sigma}} \|u\|_{L^\infty(S_{2\sigma})} + C\sqrt{\sigma} \left(\int_{S_{2\sigma}} |f|^q dx \right)^{1/q},$$

where $C = C(n, p, q) > 0$. Indeed, we consider

$$\tilde{\phi}(y) := \frac{1}{2\sigma} [\phi((2\sigma)^{1/2}A^{-1}y) - \phi(0) - 2\sigma]$$

for all $y \in (2\sigma)^{-1/2}AS_{2\sigma}(\phi) = S_1(\tilde{\phi})$, where A is from Lemma 2.1 satisfying $\det A = 1$, $\|A^{-1}\| \leq c_1^{-1/2}$ and $\|A\| \leq c_2^{1/2}$ for two universal constants. Denote

$$\tilde{u}(y) = u((2\sigma)^{1/2}A^{-1}y) \quad \text{and} \quad \tilde{f}(y) = 2\sigma f((2\sigma)^{1/2}A^{-1}y).$$

Applying Lemma 2.8 to $\tilde{u}$ in $S_1(\tilde{\phi})$, we obtain that

$$\left(\int_{S_{1/2}(\tilde{\phi})} |D\tilde{u}|^p dy \right)^{1/p} \leq C\|\tilde{u}\|_{L^\infty(S_1(\tilde{\phi}))} + C \left(\int_{S_1(\tilde{\phi})} |\tilde{f}|^q dy \right)^{1/q}.$$

Then we scale it back to obtain (3.1).

Since $u - l$ solves the same equation as u , we may apply (3.1) to $u - l$ to obtain

$$\left(\int_{S_\sigma} |D(u - l)|^p dx \right)^{1/p} \leq \frac{C}{\sqrt{\sigma}} \|u - l\|_{L^\infty(S_{2\sigma})} + C\sqrt{\sigma} \left(\int_{S_{2\sigma}} |f|^q dx \right)^{1/q}.$$

Then

$$\begin{aligned} E_p(S_\sigma) &= \left(\int_{S_\sigma} |Du - (Du)_{S_\sigma}|^p dx \right)^{1/p} \leq 2 \left(\int_{S_\sigma} |D(u - l)|^p dx \right)^{1/p} \\ &\leq \frac{C}{\sqrt{\sigma}} (\|u - v\|_{L^\infty(S_{2\sigma})} + \|v - l\|_{L^\infty(S_{2\sigma})}) + C\sigma^{\frac{1}{2} - \frac{n}{2q}} \left(\int_{S_1} |f|^q dx \right)^{1/q} \\ &\leq C \left(\varepsilon\sigma^{-1/2} + M\sigma^{1/2} + \sigma^{\frac{q-n}{2q}} \delta \right). \end{aligned}$$

We first choose σ sufficiently small such that $CM\sigma^{1/2} \leq c_0/3$, then choose ε sufficiently small such that $C\varepsilon\sigma^{-1/2} \leq c_0/3$. Finally we choose $\delta \leq \min\{\delta_1, \delta_2, \delta_3\}$ small enough such that $C\delta\sigma^{\frac{q-n}{2q}} \leq c_0/3$. In conclusion, we obtain $E_p(S_\sigma) \leq c_0$. $\square$

Theorem 3.2. *Let Ω be a normalized convex domain in $\mathbb{R}^n$. Let $\phi \in C^2(\Omega)$ be a strictly convex function in Ω with $\phi = 0$ on $\partial\Omega$. Let x_0 be the minimum point of ϕ and a Lebesgue point of Du . Let $f \in L^q(\Omega)$ with $\frac{n}{2} < q \leq n$ and let u be a solution of (1.1). Choose p so that $n < p < \frac{nq}{n-q}$ if $q < n$, and $n < p < \infty$ if $q = n$. Then for any*

$0 < \theta < 1$ and $\frac{1}{2} < \alpha < 1$, there exist $C > 0$, $\delta \in (0, 1)$ and $h_0 > 0$, all depending only on n , α , θ , p and q such that if

$$1 - \delta \leq \det D^2\phi \leq 1 + \delta \quad \text{in } \Omega,$$

then

$$(3.2) \quad |Du(x_0)| \leq C \int_{S_\phi(x_0, h)} |Du(x)| dx + C \left(\int_{S_\phi(x_0, h^\theta)} |Du(x)|^p dx \right)^{1/p} + C \mathbf{I}_{\phi, q}^{f, \alpha}(x_0, h^\theta)$$

whenever $S_\phi(x_0, h^\theta) \Subset \Omega$ with $h \leq h_0$.

Proof. For convenience, we assume that the minimum point of ϕ is $x_0 = 0$. Let δ and σ be as in Lemma 3.1.

Now we claim that there exists a sequence of positive definite matrices $\{A_j\}_{j=0}^\infty$ with $A_0 = I$ and $\det A_j = 1$ such that for all $j = 1, 2, 3, \dots$,

$$\begin{aligned} \|A_{j-1}A_j^{-1}\| &\leq \frac{1}{\sqrt{c_1}}, \\ \|A_j\| &\leq \sqrt{c_2(1 + Ca_0)(1 + Ca_1) \cdots (1 + Ca_{j-1})}, \\ \|A_j^{-1}\| &\leq 1/\sqrt{c_1(1 - Ca_1) \cdots (1 - Ca_{j-1})}, \\ B_{(1-a_j)\sqrt{2}}(0) &\subset \sigma^{-\frac{j}{2}}A_j S_{\sigma^j}(\phi) \subset B_{(1+a_j)\sqrt{2}}(0), \end{aligned}$$

where C, c_1, c_2 are the same constants as in Lemma 2.1 and the sequence $\{a_j\}_{j=0}^\infty$ is defined by

$$\begin{aligned} a_0 &:= 0, \quad a_1 := C(\sigma^{1/2} + \sigma^{-1}\delta^{1/2}), \\ a_j &:= C(a_{j-1}\sigma^{1/2} + \sigma^{-1}\delta^{1/2}) \quad \text{for all } j \geq 2. \end{aligned}$$

The claim above is proved by induction similarly to [GN1, Theorem 4.5]. For convenience, we indicate the proof for the cases $j = 1, 2$.

$j = 1$: Applying Lemma 2.1, we obtain a positive definite matrix $M = A^t A$ with $\det M = 1$, $c_1 I \leq M \leq c_2 I$ such that if we take $A_1 := A$ then

$$B_{(1-a_1)\sqrt{2}}(0) \subset \sigma^{-\frac{1}{2}}A_1 S_\sigma(\phi) \subset B_{(1+a_1)\sqrt{2}}(0)$$

with

$$a_1 := C(\sigma^{1/2} + \sigma^{-1}\delta^{1/2}) < 1 - \frac{6}{5\sqrt{2}} < \frac{1}{4}.$$

Since $\det M = 1$, $c_1 I \leq M \leq c_2 I$, we have $\|A_1^{-1}\| \leq 1/\sqrt{c_1}$ and $\|A_1\| \leq \sqrt{c_2}$.

$j = 2$: Let $\Omega_1 := \sigma^{-\frac{1}{2}}A_1 S_\sigma(\phi)$ and

$$\phi_1(y) := \frac{1}{\sigma} \left(\phi(\sigma^{\frac{1}{2}}A_1^{-1}y) - \phi(0) - \sigma \right), \quad \forall y \in \Omega_1.$$

Then we have $\Omega_1 = S_1(\phi_1)$ and

$$\begin{cases} 1 - \delta \leq \det D^2\phi_1 \leq 1 + \delta & \text{in } \Omega_1 \\ \phi_1 = 0 & \text{on } \partial\Omega_1. \end{cases}$$

Applying Lemma 2.2 for ϕ_1 and Ω_1 , we obtain a positive definite matrix $M = A^t A$ satisfying $\det M = 1$ and $(1 - Ca_1)I \leq M \leq (1 + Ca_1)I$, such that

$$B_{(1-a_2)\sqrt{2}}(0) \subset \sigma^{-\frac{1}{2}} A_1 S_\sigma(\phi_1) \subset B_{(1+a_2)\sqrt{2}}(0),$$

with

$$a_2 := C(a_1 \sigma^{1/2} + \sigma^{-1} \delta^{1/2}).$$

Define $A_2 = AA_1$. Then $\det A_2 = 1$. As $S_\sigma(\phi_1) = \sigma^{-\frac{1}{2}} A_1 S_{\sigma^2}(\phi)$, we conclude that

$$B_{(1-a_2)\sqrt{2}}(0) \subset \sigma^{-1} A_2 S_{\sigma^2}(\phi) \subset B_{(1+a_2)\sqrt{2}}(0).$$

Since $(1 - Ca_1)I \leq M \leq (1 + Ca_1)I$ and $A_1 A_2^{-1} = A^{-1}$, we have

$$\|A_1 A_2^{-1}\| = \|A^{-1}\| \leq \frac{1}{\sqrt{1 - Ca_1}} \leq \frac{1}{\sqrt{c_1}}.$$

By $A^t A = M$, we have

$$(1 - Ca_1)|x|^2 \leq |Ax|^2 \leq (1 + Ca_1)|x|^2.$$

Then

$$|A_2 x|^2 = |AA_1 x|^2 \leq (1 + Ca_1)|A_1 x|^2 \leq c_2(1 + Ca_1)|x|^2,$$

which yields $\|A_2\| \leq \sqrt{c_2(1 + Ca_1)}$. Similarly, we have $\|A_2^{-1}\| \leq \frac{1}{\sqrt{c_1(1 - Ca_1)}}$.

Now we estimate $\|A_j\|$ and $\|A_j^{-1}\|$. Choose σ so small that $C\sqrt{\sigma} < 1/2$ and then choose δ so small that $Ca_j < 1/2$ for all j . The recursion for a_j gives, for all $j \geq 1$,

$$a_j \leq (C\sqrt{\sigma})^j + \frac{2C\sqrt{\delta}}{\sigma}.$$

It follows that for all $j \geq 2$,

$$\begin{aligned} \prod_{i=1}^{j-1} (1 + Ca_i) &= \exp \left(\sum_{i=1}^{j-1} \log(1 + Ca_i) \right) \leq \exp \left(C \sum_{i=1}^{j-1} a_i \right) \\ &\leq \exp \left(C \sum_{i=1}^{j-1} (C\sqrt{\sigma})^i + \frac{2(j-1)C\sqrt{\delta}}{\sigma} \right) \leq C \exp \left(\frac{2jC\sqrt{\delta}}{\sigma} \right). \end{aligned}$$

Then we have

$$\|A_j\| \leq \left[c_2 \prod_{i=1}^{j-1} (1 + Ca_i) \right]^{\frac{1}{2}} \leq C \exp \left(\frac{jC\sqrt{\delta}}{\sigma} \right),$$

while $\|A_j^{-1}\|$ satisfies a similar estimate. So we have

$$(3.3) \quad \|A_j\| + \|A_j^{-1}\| \leq CL^j,$$

where

$$L = \exp(2C\sqrt{\delta}/\sigma).$$

Now we prove (3.2). For convenience, we denote $h_j := \sigma^j$ and $S_j := S_{h_j}(\phi)$ for all $j \geq 1$. Let $\Omega_j := h_j^{-1/2} A_j S_j$. Consider

$$\phi_j(y) := \frac{1}{h_j} \left(\phi(h_j^{1/2} A_j^{-1} y) - \phi(0) - h_j \right), \quad u_j(y) := \frac{u(h_j^{1/2} A_j^{-1} y)}{M_j}, \quad \forall y \in \Omega_j,$$

where

$$M_j := h_j^{1/2} \left(\int_{S_j} |Du(x) A_j^{-1} - (Du A_j^{-1})_{S_j(\phi)}|^p dx \right)^{1/p} + \frac{h_j}{\delta} \left(\int_{S_j} |f(x)|^q dx \right)^{1/q} + \varepsilon_0,$$

for a small parameter ε_0 . Then for $\Omega_j = S_1(\phi_j)$, we have

$$\begin{cases} 1 - \delta \leq \det D^2 \phi_j \leq 1 + \delta & \text{in } \Omega_j, \\ \phi_j = 0 & \text{on } \partial \Omega_j, \end{cases}$$

and

$$\text{tr}(\Phi_j D^2 u_j) = \frac{h_j f(h_j^{1/2} A_j^{-1} y)}{M_j} =: f_j(y)$$

in Ω_j , where $\Phi_j := \det D^2 \phi_j(y) (D^2 \phi_j)^{-1}(y)$.

Noting that

$$\begin{aligned} E_p(S_1(\phi_j)) &:= \left(\int_{S_1(\phi_j)} |Du_j - (Du_j)_{S_1(\phi_j)}|^p dy \right)^{1/p} \\ &= \frac{h_j^{1/2}}{M_j} \left(\int_{S_j} |Du(x) A_j^{-1} - (Du A_j^{-1})_{S_j(\phi)}|^p dx \right)^{1/p} \leq 1 \end{aligned}$$

and

$$\left(\int_{S_1(\phi_j)} |f_j(y)|^q dy \right)^{1/q} = \frac{h_j}{M_j} \left(\int_{S_j} |f(x)|^q dx \right)^{1/q} \leq \delta,$$

we can apply Lemma 3.1 to obtain

$$\begin{aligned} c_0 &\geq E_p(S_\sigma(\phi_j)) := \left(\int_{S_\sigma(\phi_j)} |Du_j - (Du_j)_{S_\sigma(\phi_j)}|^p dy \right)^{1/p} \\ &= \frac{h_j^{1/2}}{M_j} \left(\int_{S_{j+1}(\phi)} |Du(x) A_j^{-1} - (Du A_j^{-1})_{S_{j+1}(\phi)}|^p dx \right)^{1/p}. \end{aligned}$$

For convenience, we write

$$\mathcal{E}_j := \left(\int_{S_j} |Du(x)A_j^{-1} - (Du A_j^{-1})_{S_j}|^p dx \right)^{1/p}.$$

Letting $\varepsilon_0 \rightarrow 0$, we have

$$(3.4) \quad \begin{aligned} \mathcal{E}_{j+1} &\leq \|A_j A_{j+1}^{-1}\| \left(\int_{S_{j+1}} |Du(x)A_j^{-1} - (Du A_j^{-1})_{S_{j+1}}|^p dx \right)^{1/p} \\ &\leq \frac{c_0}{\sqrt{c_1}} h_j^{-1/2} M_j = \frac{c_0}{\sqrt{c_1}} \mathcal{E}_j + \frac{c_0 \sqrt{h_j}}{\sqrt{c_1} \delta} \left(\int_{S_j} |f(x)|^q dx \right)^{1/q}. \end{aligned}$$

Given $0 < h \ll 1$, there exists $k \geq 2$ such that

$$h_k < h \leq h_{k-1},$$

so that $S_k \subset S_\phi(x_0, h) \subset S_{k-1}$ and the two sections have comparable volume. Given $0 < \theta < 1$, let

$$N = \lfloor \theta k \rfloor$$

and denote

$$E_j := \left(\int_{S_j} |Du - (Du)_{S_j}|^p dx \right)^{1/p}, \quad \rho := \frac{c_0}{\sqrt{c_1}} \quad \text{and} \quad F_j := \left(\int_{S_j} |f(x)|^q dx \right)^{1/q}.$$

Then

$$E_j \leq \|A_j\| \mathcal{E}_j, \quad \mathcal{E}_j \leq \|A_j^{-1}\| E_j.$$

We choose the constants in the following order. First choose c_0 small in Lemma 3.1; this fixes a corresponding σ . Then shrink δ so that $L = \exp(2C\sqrt{\delta}/\sigma)$ is sufficiently close to 1. With this choice we have

$$(3.5) \quad L^{1+\theta} \rho^{1-\theta} < 1, \quad L \sigma^{(1-\theta)(\alpha-\frac{1}{2})} \leq 1, \quad L \rho < 1.$$

Iterating (3.4) from N to i , we have

$$(3.6) \quad E_i \leq CL^i \mathcal{E}_i \leq CL^i \left(\rho^{i-N} \mathcal{E}_N + \sum_{j=N}^{i-1} \rho^{i-1-j} h_j^{1/2} F_j \right).$$

Then

$$\begin{aligned} \sum_{i=k}^{\infty} E_i &\leq C \sum_{i=k}^{\infty} L^{i+N} \rho^{i-N} E_N + C \sum_{i=k}^{\infty} \sum_{j=N}^{i-1} L^i \rho^{i-1-j} h_j^{1/2} F_j \\ &\leq CL^{N+k} \rho^{k-N} E_N + C \sum_{j=N}^{k-1} L^k \rho^{k-1-j} h_j^{1/2} F_j + C \sum_{j=k}^{\infty} L^{j+1} h_j^{1/2} F_j. \end{aligned}$$

The choice of c_0 , δ and σ in (3.5) ensures that

$$L^{N+k} \rho^{k-N} \leq C(L^{1+\theta} \rho^{1-\theta})^k \leq C,$$

$$\begin{aligned}
L^k \rho^{k-1-j} h_j^{1/2} &= L^k \rho^{k-N-1} (\rho^{-1} \sigma^{\alpha-\frac{1}{2}})^{j-N} h_N^{\alpha-\frac{1}{2}} h_j^{1-\alpha} \\
&\leq C \max\{(L\rho^{1-\theta})^k, (L\sigma^{(1-\theta)(\alpha-\frac{1}{2})})^k\} h_N^{\alpha-\frac{1}{2}} h_j^{1-\alpha} \\
&\leq C h_N^{\alpha-\frac{1}{2}} h_j^{1-\alpha}, \quad \text{for all } N \leq j < k, \\
L^{j+1} h_j^{1/2} &= L^{j+1} \sigma^{(j-N)(\alpha-\frac{1}{2})} h_N^{\alpha-\frac{1}{2}} h_j^{1-\alpha} \leq C (L\sigma^{(1-\theta)(\alpha-\frac{1}{2})})^j h_N^{\alpha-\frac{1}{2}} h_j^{1-\alpha} \\
&\leq C h_N^{\alpha-\frac{1}{2}} h_j^{1-\alpha}, \quad \text{for all } j \geq k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(3.7) \quad \sum_{i=k}^{\infty} E_i &\leq C E_N + C h_N^{\alpha-\frac{1}{2}} \sum_{j=N}^{\infty} h_j^{1-\alpha} F_j \\
&\leq C E_N + C h_N^{\alpha-\frac{1}{2}} \int_0^{h_{N-1}} \left(\int_{S_\phi(x_0, \rho)} |f(x)|^q dx \right)^{1/q} \frac{d\rho}{\rho^\alpha} \\
&\leq C \left(\int_{S_\phi(x_0, h^\theta)} |Du(x)|^p dx \right)^{1/p} + C \mathbf{I}_{\phi, q}^{f, \alpha}(x_0, h^\theta).
\end{aligned}$$

Hence,

$$\begin{aligned}
|Du(x_0)| &= \lim_{m \rightarrow \infty} |(Du)_{S_{m+1}}| \leq \sum_{i=k}^{\infty} |(Du)_{S_{i+1}} - (Du)_{S_i}| + |(Du)_{S_k}| \\
&\leq \sum_{i=k}^{\infty} \left(\int_{S_{i+1}} |Du - (Du)_{S_i}|^p dx \right)^{1/p} + |(Du)_{S_k}| \\
&\leq C \sum_{i=k}^{\infty} E_i + C \int_{S_\phi(x_0, h)} |Du(x)| dx \\
&\leq C \int_{S_\phi(x_0, h)} |Du(x)| dx + C \left(\int_{S_\phi(x_0, h^\theta)} |Du(x)|^p dx \right)^{1/p} + C \mathbf{I}_{\phi, q}^{f, \alpha}(x_0, h^\theta).
\end{aligned}$$

□

Proof of Theorem 1.1. Let x_0 be as in Theorem 1.1 and fix $\varepsilon_0 > 0$. By the properties of sections [Gu, Theorem 3.3.8] and $g \in C(\Omega)$, there exist $C_1, C_2, h_1 > 0$ depending only on $\text{dist}(\Omega', \partial\Omega)$, n, λ, Λ , the modulus of continuity of g , and $b = b(n, \lambda, \Lambda) > 0$ such that

$$B(x_0, C_1 h_1) \subset S_\phi(x_0, h_1) \subset B(x_0, C_2 h_1^b)$$

and

$$|g(y) - g(x_0)| \leq \varepsilon_0, \quad \forall y \in S_\phi(x_0, h_1).$$

By John's lemma, there exists affine transformation $T := A(x - x_0) + y_0$ such that

$$B_1(0) \subset TS_\phi(x_0, h_1) \subset B_n(0),$$

which implies

$$(3.8) \quad \|A\| \leq Ch_1^{-1}, \quad \|A^{-1}\| \leq Ch_1^b.$$

By the volume estimates of sections (see [Le5, Lemma 5.6(i)]), we have

$$(3.9) \quad C^{-1} \leq |\det A|^{\frac{2}{n}} h_1 \leq C$$

for some $C > 0$ depending only on n, λ, Λ , which means that $|\det A| \approx h_1^{-n/2}$.

Denote $\tilde{\Omega} := TS_\phi(x_0, h_1)$ and consider the functions

$$(3.10) \quad \tilde{\phi}(y) := K [\phi(T^{-1}y) - \phi(x_0) - D\phi(x_0) \cdot (T^{-1}y - x_0) - h_1],$$

$$(3.11) \quad \tilde{u}(y) := Kg(x_0)u(T^{-1}y),$$

for $y \in \tilde{\Omega}$, where $K = \frac{|\det A|^{2/n}}{g(x_0)^{1/n}}$. Noting that

$$\det D^2\tilde{\phi}(y) = \frac{g(T^{-1}y)}{g(x_0)}, \quad \text{and} \quad |g(T^{-1}y) - g(x_0)| < \varepsilon_0 \text{ in } \tilde{\Omega},$$

we have

$$1 - \frac{\varepsilon_0}{\lambda} \leq \det D^2\tilde{\phi}(y) \leq 1 + \frac{\varepsilon_0}{\lambda}.$$

Meanwhile, $\tilde{\phi} = 0$ on $\partial\tilde{\Omega}$ and

$$\tilde{\Phi}^{ij} D_{ij}\tilde{u} = \Phi^{ij}(T^{-1}y) D_{ij}u(T^{-1}y) = \tilde{f}$$

for any $y \in \tilde{\Omega} = S_{\tilde{\phi}}(y_0, Kh_1)$, where $\tilde{f}$ is defined by $\tilde{f}(y) = f(T^{-1}y)$ for any $y \in \tilde{\Omega}$.

Now choose $\varepsilon_0 = \lambda\delta$ as in Theorem 3.2 and scale back to obtain

$$\begin{aligned} |Du(x_0)| &\leq \frac{\|A\|}{Kg(x_0)} |D\tilde{u}(y_0)| \leq C |D\tilde{u}(y_0)| \\ &\leq C \int_{S_{\tilde{\phi}}(y_0, h)} |D\tilde{u}(y)| \, dy + C \left(\int_{S_{\tilde{\phi}}(y_0, h^\theta)} |D\tilde{u}(y)|^p \, dy \right)^{1/p} + C \mathbf{I}_{\tilde{\phi}, q}^{\tilde{f}, \alpha}(y_0, h^\theta) \\ &\leq C \int_{S_\phi(x_0, t)} |Du(x)| \, dx + C \left(\int_{S_\phi(x_0, t^\theta)} |Du(x)|^p \, dx \right)^{1/p} + C \mathbf{I}_{\phi, q}^{f, \alpha}(x_0, ct^\theta), \end{aligned}$$

for all $t \leq \min\{K^{-1}h_0, K^{1/\theta-1}h_1^{1/\theta}\} =: h_*$, where $C > 0$ depends only on $n, \alpha, \theta, \lambda, \Lambda, p, q, \text{dist}(\Omega', \partial\Omega)$ and the modulus of continuity of g , $c > 0$ depends only on $n, \lambda, \Lambda, \theta, \text{dist}(\Omega', \partial\Omega)$ and the modulus of continuity of g . $\square$

4. INTERIOR GRADIENT CONTINUITY VIA POTENTIALS

First, we consider the normalized domains.

Theorem 4.1. *Let $\phi \in C^2(\Omega)$ be strictly convex in a normalized domain $\Omega \subset \mathbb{R}^n$, let $f \in L^q(\Omega)$ with $\frac{n}{2} < q \leq n$, and let u be the solution to (1.1). Choose p so that*

$n < p < \frac{nq}{n-q}$ if $q < n$, and $n < p < \infty$ if $q = n$. Then for any $\frac{1}{2} < \alpha < 1$, there exist $C > 0$ and $\delta \in (0, 1)$, both depending only on n, α, p and q , such that if

$$1 - \delta \leq \det D^2\phi \leq 1 + \delta \text{ in } \Omega,$$

and

$$(4.1) \quad \lim_{h \rightarrow 0} \sup_{\substack{x \in \Omega' \\ S_\phi(x, h) \in \Omega}} \mathbf{I}_{\phi, q}^{f, \alpha}(x, h) = 0 \text{ for every } \Omega' \Subset \Omega,$$

then Du has a continuous representative. Moreover, given $\theta \in (0, 1)$, we have

$$(4.2) \quad |Du(x_1) - Du(x_2)| \leq C \|Du\|_{L^\infty(\Omega'')} |x_1 - x_2|^{\kappa\beta} + C \max_{x \in \{x_1, x_2\}} \mathbf{I}_{\phi, q}^{f, \alpha}(x, c|x_1 - x_2|^{\beta\theta})$$

for all $x_1, x_2 \in \Omega' \Subset \Omega'' \Subset \Omega$, where $C > 0$ depends only on $n, \alpha, \theta, p, q, \Omega'$ and Ω'' , $c > 0$ depends only on n, α, p and q , $\beta > 1$ depends only on n and Ω' , and $\kappa > 0$ depends only on n, p, q, α and θ .

Proof. Let $\Omega' \Subset \Omega'' \Subset \Omega$ be open subsets such that $S_\phi(x_0, h_0) \Subset \Omega''$ for all $x_0 \in \Omega'$. By (3.2) in Theorem 3.2 and a standard covering argument, we obtain

$$(4.3) \quad \|Du\|_{L^\infty(\Omega'')} \leq M$$

for some constant $M > 0$ depending on $n, p, q, \text{dist}(\Omega'', \partial\Omega), \sup_{\substack{x \in \Omega' \\ S_\phi(x, h_0) \in \Omega''}} \mathbf{I}_{\phi, q}^{f, \alpha}(x, h_0)$

and $\|Du\|_{L^p(\Omega)}$.

We first show that $\{(Du)_{S_\phi(x_0, t)}\}_{t \leq h_0}$ is a Cauchy family uniformly in $x_0 \in \Omega'$. We use the notation from the proof of Theorem 3.2; in this paragraph the decay factor is denoted by $\rho = c_0/\sqrt{c_1}$, while r, t denote section heights. Thus, taking $S_k = S_\phi(x_0, \sigma^k)$, $h_k \leq h < h_{k-1}$, and $N = \lfloor \theta k \rfloor$, and arguing similarly to (3.7), for $m > k > 0$, we have

$$\begin{aligned} |(Du)_{S_{m+1}} - (Du)_{S_k}| &\leq \sum_{i=k}^m |(Du)_{S_{i+1}} - (Du)_{S_i}| \leq \sum_{i=k}^\infty E_i \\ &\leq C(L^{1+\theta} \rho^{1-\theta})^k E_N + C \mathbf{I}_{\phi, q}^{f, \alpha}(x_0, h^\theta) \\ &\leq C h_k^\kappa \|Du\|_{L^\infty(\Omega'')} + C \mathbf{I}_{\phi, q}^{f, \alpha}(x_0, h^\theta), \end{aligned}$$

where

$$\kappa := -\frac{\log(L^{1+\theta} \rho^{1-\theta})}{\log(1/\sigma)} > 0.$$

Consequently, for any $0 < r < t \leq h_0$,

$$|(Du)_{S_\phi(x_0, r)} - (Du)_{S_\phi(x_0, t)}| \leq C t^\kappa \|Du\|_{L^\infty(\Omega'')} + C \mathbf{I}_{\phi, q}^{f, \alpha}(x_0, t^\theta).$$

Using (4.1), the averages converge uniformly in $x_0 \in \Omega'$. Denote the uniform limit by

$$V(x_0) := \lim_{r \rightarrow 0} (Du)_{S_\phi(x_0, r)}.$$

Then

$$(4.4) \quad |V(x_0) - (Du)_{S_\phi(x_0, t)}| \leq C t^\kappa \|Du\|_{L^\infty(\Omega'')} + C \mathbf{I}_{\phi, q}^{f, \alpha}(x_0, t^\theta).$$

For each fixed $t > 0$, Lemma 2.3 implies that

$$x_0 \mapsto (Du)_{S_\phi(x_0, t)}$$

is continuous. The convergence in (4.4) is uniform, hence V is continuous. Moreover, $V = Du$ at every Lebesgue point of Du , and we henceforth use V as the continuous representative of Du in Ω' .

Finally, it remains to estimate the modulus of continuity of Du . We first consider $x_1, x_2 \in \Omega'$ such that $x_2 \in \partial S_\phi(x_1, \tau)$ with $\tau \leq h_0$. The engulfing property of sections (see [Le5, Theorem 5.28]) yields $S_\phi(x_1, \tau) \subset S_\phi(x_2, \theta_0 \tau)$ for some $\theta_0 = \theta_0(n, p, q, \alpha) > 0$. So we have

$$\begin{aligned} |Du(x_1) - Du(x_2)| &\leq |Du(x_1) - (Du)_{S_\phi(x_1, \tau)}| + |Du(x_2) - (Du)_{S_\phi(x_2, \theta_0 \tau)}| \\ &\quad + |(Du)_{S_\phi(x_1, \tau)} - (Du)_{S_\phi(x_2, \theta_0 \tau)}| =: \text{I} + \text{II} + \text{III}. \end{aligned}$$

By (4.4), we have

$$\text{I} + \text{II} \leq C\tau^\kappa \|Du\|_{L^\infty(\Omega'')} + C\mathbf{I}_{\phi, q}^{f, \alpha}(x_1, \tau^\theta) + C\mathbf{I}_{\phi, q}^{f, \alpha}(x_2, (\theta_0 \tau)^\theta).$$

The third term

$$\begin{aligned} \text{III} &\leq \int_{S_\phi(x_1, \tau)} |Du(y) - (Du)_{S_\phi(x_2, \theta_0 \tau)}| dy \\ &\leq C \int_{S_\phi(x_2, \theta_0 \tau)} |Du(y) - (Du)_{S_\phi(x_2, \theta_0 \tau)}| dy \leq CE_p(S_\phi(x_2, \theta_0 \tau)). \end{aligned}$$

Since there exists $k \in \mathbb{N}$ such that $h_k < \theta_0 \tau \leq h_{k-1}$, similarly to (3.7) we obtain

$$\text{III} \leq CE_k \leq C \sum_{i=k}^{\infty} E_i \leq C\tau^\kappa \|Du\|_{L^\infty(\Omega'')} + C\mathbf{I}_{\phi, q}^{f, \alpha}(x_2, (\theta_0 \tau)^\theta).$$

Therefore, we obtain that

$$|Du(x_1) - Du(x_2)| \leq C\tau^\kappa \|Du\|_{L^\infty(\Omega'')} + C\mathbf{I}_{\phi, q}^{f, \alpha}(x_1, \tau^\theta) + C\mathbf{I}_{\phi, q}^{f, \alpha}(x_2, c\tau^\theta).$$

By the $C^{1, \alpha}$ estimate for the Monge-Ampère equation (see [Le5, Theorem 5.18]), we see that

$$\tau = \phi(x_2) - \phi(x_1) - D\phi(x_1) \cdot (x_2 - x_1) \leq C|x_1 - x_2|^\beta,$$

for some $\beta = \beta(n, p, q, \alpha) > 1$ and $C = C(n, p, q, \alpha, \text{diam}(\Omega'))$. Hence, we have

$$|Du(x_1) - Du(x_2)| \leq C\|Du\|_{L^\infty(\Omega'')} |x_1 - x_2|^{\kappa\beta} + C \max_{x \in \{x_1, x_2\}} \mathbf{I}_{\phi, q}^{f, \alpha}(x, c|x_1 - x_2|^{\theta\beta}).$$

If $x_2 \notin S_\phi(x_1, \tau)$ for all $\tau \leq h_0$, then we have

$$h_0 \leq \phi(x_2) - \phi(x_1) - D\phi(x_1) \cdot (x_2 - x_1) \leq C|x_1 - x_2|^\beta.$$

So we have

$$|Du(x_1) - Du(x_2)| \leq 2 \sup_{\Omega''} |Du| \leq Ch_0^{-\kappa} |x_1 - x_2|^{\kappa\beta} \|Du\|_{L^\infty(\Omega'')}.$$

Therefore, we obtain (4.2).

□

Proof of Theorem 1.4. Using the same scaling in (3.10) and (3.11) as in the proof of Theorem 1.1, one can obtain

$$\begin{aligned}
|Du(x_1) - Du(x_2)| &\leq C|D\tilde{u}(y_1) - D\tilde{u}(y_2)| \\
&\leq C\|Du\|_{L^\infty(\Omega'')}|y_1 - y_2|^{\kappa\beta} + C \max_{y \in \{y_1, y_2\}} \mathbf{I}_{\phi, q}^{\tilde{f}, \alpha}(y, \tilde{c}|y_1 - y_2|^{\theta\beta}) \\
&\leq C\|Du\|_{L^\infty(\Omega'')}|x_1 - x_2|^{\kappa\beta} + C \max_{x \in \{x_1, x_2\}} \mathbf{I}_{\phi, q}^{f, \alpha}(x, c|x_1 - x_2|^{\theta\beta}),
\end{aligned}$$

where $c > 0$ depends only on $n, p, q, \alpha, \lambda, \Lambda, \Omega'$ and the modulus of continuity of g , and $C > 0$ depends only on $n, \lambda, \Lambda, p, q, \theta, \alpha, \Omega$ and the modulus of continuity of g . □

Proof of Corollary 1.5. We present the argument in a normalized section; the general local statement follows by the affine normalization employed in the proof of Theorem 1.1. All constants below are local constants; after normalization they are uniform on compact subsets. Fix $\theta \in (0, 1)$ and choose $p > n$ within the admissible range. The excess-decay iteration from the proof of Theorem 3.2 yields that, for every $\Omega' \Subset \Omega'' \Subset \Omega$, there exist constants $C, \kappa > 0$, depending only on n, λ, Λ , and Ω'' , such that for all sufficiently small sections $S_\phi(x, h) \Subset \Omega''$,

$$E_p(S_\phi(x, h)) \leq Ch^\kappa \|Du\|_{L^p(\Omega'')} + C\omega_{\alpha, q}(h^\theta, \Omega').$$

Since

$$\text{mosc}_{S_\phi(x, h)} Du \leq CE_p(S_\phi(x, h)),$$

it follows immediately that $Du \in VMO_{loc}(\Omega, \phi)$ whenever $\omega_{\alpha, q}(r, \Omega') \rightarrow 0$ as $r \rightarrow 0$, and $Du \in BMO_{loc}(\Omega, \phi)$ whenever $\omega_{\alpha, q}$ remains locally bounded.

Assume now that

$$M := \sup_{S_\phi(x_0, h) \Subset \Omega} h^{\frac{1-\varepsilon}{2}} \left(\int_{S_\phi(x_0, h)} |f(y)|^n dy \right)^{1/n} < \infty$$

for some $\varepsilon \in (0, 1)$. Choose $1/2 < \alpha < (1 + \varepsilon)/2$. Then for all sufficiently small h ,

$$\mathbf{I}_{\phi, n}^{f, \alpha}(x, h) \leq Mh^{\alpha - \frac{1}{2}} \int_0^h \rho^{-\frac{1-\varepsilon}{2}} \frac{d\rho}{\rho^\alpha} \leq CMh^{\varepsilon/2}.$$

Theorem 4.1 therefore yields a local Hölder modulus for Du , hence $u \in C_{loc}^{1, \gamma}(\Omega)$ for some $\gamma > 0$. Specifically, using the modulus of continuity estimate (4.2) in Theorem 4.1 and combining it with the estimate (3.4) in [LN], we obtain

$$\begin{aligned}
|Du(y_1) - Du(y_2)| &\leq C(\|Du\|_{L^p(\Omega'')} + M)|y_1 - y_2|^{\min\{\kappa\beta, \beta\theta\varepsilon/2\}} \\
&\leq C(\|u\|_{L^\infty(\Omega)} + M)|y_1 - y_2|^{\min\{\kappa\beta, \beta\theta\varepsilon/2\}},
\end{aligned}$$

for any $y_1, y_2 \in \Omega'$, which is consistent with the result in [GN1].

Finally, if $f \in L^{s,\infty}(\Omega)$ with $s > n$, then for every $q \leq n < s$, by the Hölder's inequality in Lorentz spaces (2.1), we obtain that

$$\left(\int_{S_\phi(x,h)} |f|^q dx \right)^{1/q} \leq C \|f\|_{L^{s,\infty}(S_\phi(x,h))} |S_\phi(x,h)|^{\frac{1}{q}-\frac{1}{s}} \leq C \|f\|_{L^{s,\infty}(S_\phi(x,h))} h^{\frac{n}{2q}-\frac{n}{2s}}.$$

Taking $\alpha = 1 - \frac{n}{2s} > \frac{1}{2}$ gives a uniform bound for $\omega_{\alpha,q}$ and $Du \in BMO_{\text{loc}}(\Omega, \phi)$.

If $f \in L^r(\Omega)$ for some $r > n$, we set $\varepsilon := 1 - n/r > 0$. By Hölder's inequality, we have

$$h^{\frac{1-\varepsilon}{2}} \left(\int_{S_\phi(x,h)} |f(y)|^n dy \right)^{1/n} = Ch^{\frac{n-r}{2r}} \|f\|_{L^n(S_\phi(x,h))} \leq C \|f\|_{L^r(\Omega)} < \infty,$$

which yields that u is $C^{1,\gamma}$ locally. $\square$

Acknowledgments. The author would like to thank his PhD supervisor, Prof. Bin Zhou, for his constant encouragement and helpful suggestions. The author would also like to thank Dr. Ling Wang for his interest and helpful comments. The author is also grateful to Prof. Nam Q. Le for his valuable comments and generous suggestions.

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