NOTES ON MONOTONICITY APPROACH

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This notes contain two parts, including Yuan's Pogorelov's Hessian estimates for Monge-Ampère equation [Yuan], Collins and Tong's boundary regularity of optimal transport map on convex domains[CT]. The same point is that they all use the monotonicity approach.

1. Monotonicity approach to Pogorelov's Hessian estimates

Theorem 1.1. [Yuan] *Let u be a smooth convex solution to*

$$\det D^2 u = 1$$

with $Du(0) = 0$ on

$$D_r := \left\{x \in \mathbb{R}^n : x \cdot \nabla u \leq r^2\right\}.$$

Then we have

$$|D^2 u(0)| \leq \left[2|B_1| r^n \frac{|\partial D_r|}{|D_r|^2} \|Du\|_{L^\infty(D_r)}\right]^{2n}. \tag{1.1}$$

Remark 1.2. *We define the following sets*

$$D_r(u, x_0) := \{x \in \Omega : (x - x_0) \cdot (\nabla u(x) - \nabla u(x_0)) \leq r^2\}$$

which we call **extrinsic balls of radius r centered at x_0**. *Up to a factor of 2, the extrinsic ball and the sections have the same shape. Indeed, we assume that $x_0 = 0$, $u(0) = 0$, and $\nabla u(0) = 0$ by subtracting a linear function, by convexity we have*

$$u(x) = u(x) - u(0) \leq x \cdot \nabla u(x) \leq u(2x) - u(x) \leq u(2x), \tag{1.2}$$

from which the two inclusions follows immediately

$$\frac{1}{2} S_{r^2}(u, x_0) \subset D_r(u, x_0) \subset S_{r^2}(u, x_0)$$

Proof of Theorem 1.1. Taking the gradient of both sides of $\log \det D^2 u = 0$, we have

$$u^{ij} u_{ijk} = 0, \quad \text{for each } k = 1, 2, \cdots, n. \tag{1.3}$$

Denote $g = (g_{ij}) = D^2 u$, and we can write the Laplace-Beltrami operator as $\Delta_g = g^{ij} \partial_{ij}$. Let

$$z = x \cdot \nabla u(x) = x \cdot (\nabla u(x) - \nabla u(0)) \geq 0.$$

For any fixed point p, since the object we study is invariant under coordinate rotation, we may assume that $D^2 u$ is diagonal (which yields $g^{ii} = \frac{1}{u_{ii}}$ at the point p), then at the point $x = p$, we have

$$\begin{aligned}
|\nabla_g z|^2 &= \left\langle z_i g^{ij} \frac{\partial}{\partial x_j}, z_k g^{kl} \frac{\partial}{\partial x_l} \right\rangle_g = g^{ij} \partial_i z \partial_j z = g^{ij} (u_i + x_k u_{ki})(u_j + x_k u_{kj}) \\
&\overset{p}{=} g^{ii} (u_i^2 + x_i^2 u_{ii}^2 + 2 x_i u_i u_{ii}) = \frac{u_i^2}{u_{ii}} + x_i^2 u_{ii} + 2 x_i u_i \geq 4 x_i u_i = 4 x \cdot \nabla u(x) = 4z,
\end{aligned} \tag{1.4}$$

$$\Delta_g z = g^{ij} (2 u_{ij} + x_k u_{kij}) \overset{(1.3)}{=} 2 g^{ij} u_{ij} \overset{p}{=} 2 g^{ii} u_{ii} = 2n \leq \frac{n}{2} \frac{|\nabla_g z|^2}{z}. \tag{1.5}$$

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Denote $s=\sqrt{z}$, then at the point p, we have

$$|\nabla_g s| = \frac{1}{2\sqrt{z}}|\nabla_g z| \overset{(1.4)}{\geq} 1 \tag{1.6}$$

$$\begin{aligned}
\Delta_g s = g^{ij}\partial_{ij}s = g^{ij}\left(\frac{z_{ij}}{2s} - \frac{1}{4}\frac{z_i z_j}{s^3}\right) &= \frac{1}{2s}\Delta_g z - \frac{1}{4s^3}|\nabla_g z|^2 \\
\overset{(1.5)}{\leq} \frac{1}{2s}\frac{n|\nabla_g z|^2}{2s^2} - \frac{1}{4s^3}|\nabla_g z|^2 &= \frac{n-1}{4s^3}|\nabla_g z|^2 = \frac{n-1}{s}|\nabla_g s|^2. \qquad (1.7)
\end{aligned}$$

Step 1. To show that: For any nonnegative superharmonic quanlity q, i.e. satisfying

$$q \geq 0 \quad \text{and} \quad \Delta_g q \leq 0,$$

one has

$$\frac{1}{|B_1(0)|r^n}\int_{D_r} q\,\mathrm{d}x \leq q(0). \tag{1.8}$$

Now we shall prove (1.8). Set

$$\psi(s) := \int_s^\infty t\chi\left(\frac{t}{\rho}\right)dt = \begin{cases} \frac{1}{2}\rho^2 - \frac{1}{2}s^2, & 0 \leq s \leq \rho \\ 0, & s > \rho \end{cases}$$

where χ is taken as a nonnegative smooth approximation of the characteristic function of $(-\infty,1) \subset (-\infty,\infty)$ with support in $(-\infty,1)$. Then we have

$$\begin{aligned}
\Delta_g\psi(s) &= g^{ij}(\psi' s_i)_j = \psi' g^{ij}s_{ij} + \psi'' g^{ij}s_i s_j = \psi'\Delta_g s + \psi''|\nabla_g s|^2 \\
\text{compute } \psi'\ \psi'' \quad &= -s\chi(s/\rho)\Delta_g s + \left[-\chi(s/\rho) - \frac{s}{\rho}\chi'(s/\rho)\right]|\nabla_g s|^2 \\
\text{using (1.7)} \quad &\geq -\left[n\chi(s/\rho) + \frac{s}{\rho}\chi'(s/\rho)\right]|\nabla_g s|^2 \\
&= \rho^{n+1}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^{-n}\chi(s/\rho)\right)|\nabla_g s|^2.
\end{aligned}$$

Multiplying q both sides and integrating over the whole maximal surface $M := (x, Du(x))$, we have

$$0 \geq \int_M \psi\Delta_g q\,\mathrm{d}V_g = \int_M q\Delta_g\psi\,\mathrm{d}V_g \geq \rho^{n+1}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\int_M q\rho^{-n}\chi(s/\rho)|\nabla_g s|^2\,\mathrm{d}V_g.\right]$$

Note that $1 \leq |\nabla_g s| \to 1$ as $x \to 0$ and $\mathrm{d}V_g = \sqrt{\det D^2 u}\mathrm{d}x = \mathrm{d}x$. Indeed, near 0, we denote $u_{ij}(0) = \lambda_i(0)\delta_{ij}$, $u_i = \lambda_i(0)x_i + O(|x|^2)$, $u_{ii} = \lambda_i(0) + O(|x|)$. We have

$$\begin{aligned}
|\nabla_g s|^2 = \frac{|\nabla_g z|^2}{4z} &= \frac{\frac{u_i^2}{u_{ii}} + x_i^2 u_{ii} + 2x_i u_i}{4x_i u_i} \\
&= \frac{1}{2} + \frac{\frac{\lambda_i(0)^2 x_i^2 + 2\lambda_i(0)O(|x|^3)}{\lambda_i(0)+O(|x|)} + \lambda_i(0)x_i^2 + O(|x|^3)}{4\lambda_i(0)x_i^2 + O(|x|^3)} \to \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 \quad \text{as } |x| \to 0.
\end{aligned}$$

Taking limit in the smooth approximation of the characteristic function, we know that

$$F(\rho) := \rho^{-n}\int_{\{s\leq\rho\}} q|\nabla_g s|^2\,\mathrm{d}x = \rho^{-n}\int_{\{x\cdot\nabla u(x)\leq\rho^2\}} q|\nabla_g s|^2\,\mathrm{d}x$$

is non-increasing. So $\lim_{\rho\to 0^+} F(\rho) \geq F(r)$. Since near 0, $|\{x\cdot\nabla u(x) \leq \rho^2\}| = |\{x^T D^2u(0)x \leq \rho^2\}| = \det D^2u(0)|B_\rho(0)| = |B_1(0)|\rho^n$, we have $\lim_{\rho\to 0^+} F(\rho) = q(0)|B_1(0)|$. Then

$$|B_1(0)|q(0) \geq r^{-n}\int_{D_r} q|\nabla_g s|^2\,\mathrm{d}V_g \geq r^{-n}\int_{D_r} q\,\mathrm{d}x.$$

Step 2. Jacobi inequality:

Lemma 1.3. *Suppose u is a smooth convex solution to $\det D^2u = 1$. Then*

$$\Delta_g \log\det\left(I + D^2u(x)\right) \geq \frac{1}{2n}\left|\nabla_g \log\det\left(I + D^2u(x)\right)\right|^2. \tag{1.9}$$

Or equivalently for $q(x) = [\det(I + D^2u(x))]^{-\frac{1}{2n}}$,

$$\Delta_g q \leq 0. \tag{1.10}$$

Step 3. Let

$$q(x) = \left[\det(I + D^2u(x))\right]^{-\frac{1}{2n}}$$

be the desired superharmonic quantity. Note that by Hölder inequality,

$$|D_r|^2 = \left(\int_{D_r} q^{1/2}q^{-1/2}\,\mathrm{d}x\right)^2 \leq \int_{D_r} q\,\mathrm{d}x\int_{D_r} q^{-1}\,\mathrm{d}x, \Rightarrow \frac{1}{\int_{D_r} q\,\mathrm{d}x} \leq \frac{1}{|D_r|^2}\int_{D_r} q^{-1}\,\mathrm{d}x.$$

Then we have

$$\begin{aligned}
\left(\det\left[I + D^2u(0)\right]\right)^{\frac{1}{2n}} = \frac{1}{q(0)} &\leq |B_1|r^n\frac{1}{\int_{D_r} q\,\mathrm{d}x} \qquad \text{by (1.8)}\\
&\leq |B_1|r^n\frac{1}{|D_r|^2}\int_{D_r} q^{-1}\,\mathrm{d}x\\
&= |B_1|r^n\frac{1}{|D_r|^2}\int_{D_r}\left[(1+\lambda_1)\cdots(1+\lambda_n)\right]^{\frac{1}{2n}}\,\mathrm{d}x < |B_1|r^n\frac{1}{|D_r|^2}\int_{D_r}(1+\lambda_{\max})\,\mathrm{d}x\\
&\leq |B_1|r^n\frac{1}{|D_r|^2}\int_{D_r} 2\lambda_{\max}\,\mathrm{d}x \leq |B_1|r^n\frac{2}{|D_r|^2}\int_{D_r}\Delta u\,\mathrm{d}x = |B_1|r^n\frac{2}{|D_r|^2}\int_{\partial D_r} u_\gamma\,\mathrm{d}\sigma\\
&\leq 2|B_1|r^n\frac{|\partial D_r|}{|D_r|^2}\|Du\|_{L^\infty(D_r)}.
\end{aligned}$$

Hence we obtain

$$|D^2u(0)| = \lambda_{\max}(D^2u(0)) \leq \det\left[I + D^2u(0)\right] \leq \left(2|B_1|r^n\frac{|\partial D_r|}{|D_r|^2}\|Du\|_{L^\infty(D_r)}\right)^{2n}.$$

□

Now it remains to show that Lemma 1.3.

Proof of Lemma 1.3. Denote

$$b(x) = \log\det\left(I + D^2u(x)\right).$$

For any fixed point p, we assume that D^2u is diagonalized. we can calculate at the point p

$$0 = \partial_\alpha \log\det D^2u = g^{ij}u_{ij\alpha} = g^{ii}u_{ii\alpha}, \tag{1.11}$$

$$0 = \partial_\beta\left(g^{ij}u_{ij\alpha}\right) = g^{ij}u_{ij\alpha\beta} - g^{ik}g^{lj}g_{kl\beta}u_{ij\alpha}$$

$$\Delta_g u_{\alpha\beta} = g^{ij}\partial_{ij}u_{\alpha\beta} = g^{ik}g^{lj}u_{kl\beta}u_{ij\alpha} = g^{kk}g^{ll}u_{kl\alpha}u_{kl\beta}. \tag{1.12}$$

Note that

$$\partial_\alpha b = (I+g)^{ij}u_{ij\alpha}$$

$$
\begin{aligned}
\partial_{\alpha\beta} b &= (I+g)^{ij}\partial_{\alpha\beta}u_{ij} - (I+g)^{ik}(I+g)^{lj}\partial_\beta(\delta_{kl}+g_{kl})u_{ij\alpha} \\
&= (1+u_{ii})^{-1}\partial_{\alpha\beta}u_{ii} - (1+u_{kk})^{-1}(1+u_{ll})^{-1}u_{kl\alpha}u_{kl\beta}.
\end{aligned}
$$

We obtain that

$$
\begin{aligned}
\Delta_g b &= g^{\alpha\beta}\partial_{\alpha\beta} b = g^{\alpha\alpha}\partial_{\alpha\alpha} b \\
&= (1+u_{ii})^{-1}g^{\alpha\alpha}\partial_{\alpha\alpha}u_{ii} - g^{\alpha\alpha}(1+u_{kk})^{-1}(1+u_{ll})^{-1}u_{kl\alpha}^2 \\
\text{using (1.12)} \qquad &= (1+\lambda_i)^{-1}g^{kk}g^{ll}u_{kli}^2 - g^{\alpha\alpha}(1+\lambda_k)^{-1}(1+\lambda_l)^{-1}u_{kl\alpha}^2 \\
&= (1+\lambda_i)^{-1}g^{jj}g^{kk}u_{ijk}^2 - (1+\lambda_i)^{-1}(1+\lambda_k)^{-1}g^{jj}u_{ijk}^2 \\
&= \left[\lambda_i(1+\lambda_k) - \lambda_i\lambda_k\right](1+\lambda_i)^{-1}(1+\lambda_k)^{-1}g^{ii}g^{jj}g^{kk}u_{ijk}^2 \\
&= \lambda_i(1+\lambda_i)^{-1}(1+\lambda_k)^{-1}h_{ijk}^2,
\end{aligned}
$$

where we denote the second fundamental form $\sqrt{g^{ii}g^{jj}g^{kk}}u_{ijk}$ by h_{ijk}. Let $\mu_i = \frac{\lambda_i - 1}{\lambda_i + 1} \in (-1,1)$. We have

$$
\begin{aligned}
\Delta_g b &= \frac{1+\mu_i}{1-\mu_i}\cdot\frac{1-\mu_i}{2}\cdot\frac{1-\mu_k}{2}h_{ijk}^2 = \frac{1}{4}\sum_{i,j,k}(1+\mu_i)(1-\mu_k)h_{ijk}^2 \\
&= \frac{1}{4}\left[\sum_i(1-\mu_i^2)h_{iii}^2 + \sum_{i\neq j}(1-\mu_i^2)h_{iij}^2 + \sum_{i\neq j}(1+\mu_i)(1-\mu_j)h_{ijj}^2 + \sum_{i\neq k}(1+\mu_i)(1-\mu_k)h_{iik}^2\right] \\
&\quad + \frac{1}{4}\sum_{i>j>k}\left[(2+\mu_i+\mu_j)(1-\mu_k) + (2+\mu_i+\mu_k)(1-\mu_j) + (2+\mu_k+\mu_j)(1-\mu_i)\right]h_{ijk}^2 \\
&= \frac{1}{4}\left[\sum_i(1-\mu_i^2)h_{iii}^2 + \sum_{i\neq j}(3-\mu_j^2-2\mu_i\mu_j)h_{ijj}^2\right] + \frac{1}{2}\sum_{i>j>k}(3-\mu_i\mu_j-\mu_i\mu_k-\mu_j\mu_k)h_{ijk}^2 \geq 0.
\end{aligned}
$$

At the point p, we also have

$$
\begin{aligned}
|\nabla_g b|^2 = g^{\alpha\beta}\partial_\alpha b\partial_\beta b &= \sum_\alpha g^{\alpha\alpha}\left[\sum_j \frac{1}{1+\lambda_j}u_{jj\alpha}\right]^2 \\
&= \sum_\alpha\left[\sum_j(1+\lambda_j)^{-1}\lambda_j g^{jj}\sqrt{g^{\alpha\alpha}}u_{jj\alpha}\right]^2 \\
&= \frac{1}{4}\sum_i\left[\sum_j(1+\mu_j)h_{ijj}\right]^2 = \frac{1}{4}\sum_i\left[\sum_j(1-\mu_j)h_{ijj}\right]^2,
\end{aligned}
$$

where the last inequality follows the fact that $\sum_j h_{ijj} = g^{ii}\sum_j g^{jj}u_{ijj} = 0$ since (1.11).

For each fixed i, case $\mu_i \geq 0$,

$$
\begin{aligned}
\frac{1}{2n}\left(\sum_j(1-\mu_j)h_{ijj}\right)^2 &\leq \frac{1}{2}(1-\mu_i)^2h_{iii}^2 + \frac{1}{2}\sum_{j\neq i}(1-\mu_j)^2h_{ijj}^2 \\
&\leq (1+\mu_i)(1-\mu_i)h_{iii}^2 + \sum_{j\neq i}\left[1-\mu_j^2+2(1-\mu_i\mu_j)\right]h_{ijj}^2,
\end{aligned}
$$

where in the first inequality we used

$$
a\cdot e \leq \|a\|_2\|e\|_2, \quad \text{where } a = (a_1,\cdots,a_n),\ e = (1,\cdots,1),
$$

and in the last inequality we used

$$\frac{1}{2}(1-\mu_j)^2 \leq \begin{cases} 1-\mu_j^2 & \text{for } \mu_j \in [0,1) \\ 2(1-\mu_i\mu_j) & \text{for } \mu_j \in (-1,0), \mu_i \geq 0. \end{cases}$$

For the case $\mu_i \in (-1,0)$, symmetrically we have

$$\frac{1}{2n}\left(\sum_j (1+\mu_j)h_{ijj}\right)^2 \leq (1+\mu_i)(1-\mu_i)h_{iii}^2 + \sum_{j\neq i}\left[1-\mu_j^2+2(1-\mu_i\mu_j)\right]h_{ijj}^2.$$

Hence

$$\frac{1}{2n}|\nabla_g b|^2 \leq \frac{1}{4}\sum_i\left[(1-\mu_i^2)h_{iii}^2 + \sum_{j\neq i}(3-\mu_j^2-\mu_i\mu_j)h_{ijj}^2\right] \leq \Delta_g b.$$

Equivalently, we also have

$$\begin{aligned} \Delta_g q = \Delta_g(e^{-\frac{b}{2n}}) &= g^{ij}\partial_{ij}(e^{-\frac{b}{2n}}) = g^{ij}\left(-\frac{1}{2n}e^{-\frac{b}{2n}}b_i\right)_j \\ &= \frac{1}{4n^2}e^{-\frac{b}{2n}}g^{ij}b_ib_j - \frac{1}{2n}e^{-\frac{b}{2n}}g^{ij}b_{ij} \\ &= \frac{1}{2n}e^{-\frac{b}{2n}}\left[\frac{1}{2n}|\nabla_g b|^2 - \Delta_g b\right] \leq 0. \end{aligned}$$

□

2. Preliminaries in optimal transport maps

Given domains $\Omega, \Omega' \subset \mathbb{R}^n$ equipped with measure μ, ν respectively with $\mu(\Omega) = \nu(\Omega')$. An optimal transport map (with quadratic cost) is a measurable map $T : \Omega \to \Omega'$ that satisfy $T_\sharp\mu = \nu$, i.e.

$$\int_\Omega \varphi \circ T \mathrm{d}\mu = \int_{\Omega'} \varphi \mathrm{d}\nu, \quad \text{for any } \nu\text{-integrable function}$$

and that minimizes the cost function

$$c(T) := -\int_\Omega x \cdot T(x) d\mu(x)$$

amongst all measurable maps that pushforward μ to ν.

Brenier [Br] shows that a unique optimal transport map T always exists and it is given by the gradient of a convex potential $T(x) = \nabla u(x)$ for μ-almost every x. Moreover, the inverse optimal transport map $T^{-1} = \nabla v : \Omega' \to \Omega$ is given by the gradient of the Legendre transform of u, defined by

$$v(y) := \sup_{x\in\Omega}(\langle x, y\rangle - u(x)).$$

We say

$$g'(\nabla u(x)) \det D^2u(x) = g(x)$$

in Brenier's sense if for any continuous function φ such that

$$\int_\Omega \varphi(\nabla u)g(x)\,\mathrm{d}x = \int_{\Omega'} \varphi(y)g'(y)\,\mathrm{d}y \left(= \int_\Omega \varphi(\nabla u(x))g'(\nabla u(x)) \det D^2u(x)\,\mathrm{d}x\right),$$

which is strictly weaker than the classical definition of weak solution, since it is unable to see the singular part of $\det D^2u$.

Denote that the measures $d\mu = g(x)dx$, $d\nu = g'(y)dy$ are absolutely continuous with respect to the Lebesgue measure. We present the main result on the regularity of u before:

- When Ω and Ω' are convex, and , Caffarelli [C1] proved that u satisfy the following Monge-Ampere equation in the Alexandrov sense

$$\begin{cases} \det D^2 u(x) = \frac{g(x)}{g'(\nabla u(x))} \text{ for } x \in \Omega \\ \nabla u(\Omega) = \Omega'. \end{cases} \tag{2.1}$$

  Moreover, if the domains are not convex, then Brenier's solutions can fail to be an Alexandrov solution to (2.1), and u may not even be C^1.
- If Ω, Ω' are bounded convex domains, and $C^{-1} < g, g' < C$, then $u \in C^{1,\delta}(\overline{\Omega})$ [C2]. If g, g' are allowed to degenerate, but satisfy a doubling type condition [JS], this conclusion also holds.
- If Ω, Ω' are uniformly convex with C^2 boundary, $C^{-1} < g, g' < C$, and g, g' are C^α-Hölder continuous, then $u \in C^{2,\alpha'}(\overline{\Omega})$ [C3]. Earlier works of Delanoe [De] (in dimension 2), and Urbas [Ur] established the global $C^{2,\alpha'}$ regularity assuming $\partial\Omega, \partial\Omega'$ are $C^{2,1}$, and $g, g' \in C^{1,1}$.
- Chen-Liu-Wang [CLW] removed the uniform convexity condition, proving that $u \in C^{2,\alpha}(\overline{\Omega})$ provided $C^{-1} < g, g' < C$ are C^α, and Ω, Ω' are convex and $C^{1,1}$-regular.
- Assuming Ω, Ω' are convex and $C^{1,1}$ regular, Chen-Liu-Wang[CLW] also establish the global $C^{1,1-\epsilon}$ regularity (for any $\epsilon > 0$), and $W^{2,p}$ regularity (for any $p > 0$) assuming that $C^{-1} < g, g' < C$ are continuous up to the boundary. Previously, Caffarelli established interior $W^{2,p}$ estimates in [C4], while global $W^{2,p}$ estimates were obtained for the Dirichlet problem by Savin [Sa1], and for the second boundary value problem by Chen-Figalli [CF].
- Without any regularity assumptions on the boundary, Savin-Yu [SaYu] in dimension 2 establishes the global $C^{1,1-\epsilon}$ and $W^{2,p}$ regularity of solutions when $g = g' = 1$ for arbitrary convex domains $\Omega, \Omega' \subset \mathbb{R}^2$.

In the paper[CT], The first result is a global $W^{2,p}$ and $C^{1,1-\epsilon}$ estimate for solutions of (2.1) in arbitrary convex domains in any dimension, assuming $C^{-1} \le g, g' \le C$ are C^α. This extends the results of Savin-Yu [SaYu] to all dimensions. (See Theorems 5.5 and 5.6)

The second result is a global $C^{2,\alpha}$-regularity result, which extends the result of Chen-Liu-Wang [CLW] from $C^{1,1}$ convex domains to $C^{1,\beta}$ convex domains. (See Theorem 5.9)

Definition 2.1. *An* **optimal transport map** *(or more precisely, an optimal transport map between convex domains equipped with doubling measures) will be given by the data* $((\Omega, \mu), (\Omega', \nu), u, v)$ *where*

(1) $(\Omega, \Omega') \subset (\mathbb{R}^n_x, \mathbb{R}^n_y)$ *are two open (not necessarily compact) convex sets.*

(2) μ *and* ν *are measures with support* $\overline{\Omega}$ *and* $\overline{\Omega'}$ *respectively such that*

  (a) They are absolutely continuous with respect to the Lebesgue measure, and we set $d\mu = g(x)dx$*, and* $d\nu = g'(y)dy$*.*

  (b) The measures μ *(and* ν*) are doubling, that is there exist constant* $C > 1$ *such that for any ellipsoid* $E \subset \mathbb{R}^n$ *centered at a point* $x \in \overline{\Omega}$ *(or centered at* $y \in \overline{\Omega'}$*), we have*

$$\mu(E) \le C\mu(\frac{1}{2}E) \left(\text{or } \nu(E) \le C\nu(\frac{1}{2}E) \right).$$

  The constant $C > 1$ *is called the doubling constant of* (μ, ν)*.*

(3) (u, v) *are a pair of strictly convex functions on* $(\overline{\Omega}, \overline{\Omega}')$ *such that*

  (a) $\nabla u(\Omega) = \Omega'$ *and* $\nabla v(\Omega') = \Omega$*.*

  (b) Let $(\bar{u}, \bar{v})$ *be the minimal convex extensions of* (u, v)*,*

$$\bar{u}(x) := \sup\{l(x) : l \text{ is tangent to } u \text{ at some point } x_0 \in \Omega\},$$

$$\bar{v}(y) := \sup\{l(y) : l \text{ is tangent to } v \text{ at some point } y_0 \in \Omega'\}.$$

  Then $\bar{u}, \bar{v} \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$ *for some* $\alpha \in (0,1)$*, and* $(\bar{u}, \bar{v}) = (v^\star, u^\star)$ *where* $(v^\star, u^\star)$ *are the Legendre transforms of* (v, u)*.*

(c) $(\bar{u},\bar{v})$ are Alexandrov solutions to the Monge-Ampére equations

$$\det D^2\bar{u}(x) = \frac{g(x)}{g'(\nabla\bar{u}(x))}\chi_{\Omega}(x) \tag{2.2}$$

and

$$\det D^2\bar{v}(y) = \frac{g'(y)}{g(\nabla\bar{v}(y))}\chi_{\Omega'}(y). \tag{2.3}$$

respectively.

Remark 2.2. *By* [JS, Theorem 1.1] *(also see* [SaYu, Remark 2.1]*), the properties for u, v are always satisfied by optimal transport maps when the measures (μ,ν) are doubling measures supported on convex domains.*

For any $x_0 \in \overline{\Omega}$ and $h > 0$, we define the **section of u at x_0 of height h** to be the convex set

$$S_h(u,x_0) := \{x \in \Omega : u(x) - u(x_0) - \nabla u(x_0)(x - x_0) \le h\}.$$

We also define a **centered section** of u at $x_0 \in \overline{\Omega}$ to be

$$S_h^c(u,x_0) := \{x \in \mathbb{R}^n : \bar{u}(x) - \bar{u}(x_0) - p\cdot(x - x_0) \le h\}$$

where recall that $\bar{u}$ is the minimal convex extension of u, and $p \in \mathbb{R}^n$ is chosen so that $S_h^c(u,x_0)$ has center of mass x_0. By [C3], if the graph of $\bar{u}$ contains no line, then centered sections always exist.

Given a centered section $S = S_h^c(u,x_0)$, one can define the normalized section by

$$\tilde{S} = A(S)$$

where A is the (unique) symmetric positive definite matrix such that $A(\mathcal{E}) = B_1(0)$ where $\mathcal{E}$ is the John ellipsoid of S. We call A the **normalization matrix** of S, and it satisfies $(\det A)|S| \sim 1$. We also define the normalized potential $\tilde{u} : \tilde{S} \to \mathbb{R}$ by

$$\tilde{u}(\tilde{x}) = \frac{(\bar{u} - l)(A^{-1}\tilde{x})}{h}.$$

where $l(x) = u(x_0) + p\cdot(x - x_0)$. It follows that $\tilde{S} = \{\tilde{u} < 1\}$, and the pair $(\tilde{S},\tilde{u})$ is a normalized pair. Moreover, $\nabla\tilde{u} : \tilde{\Omega} \to \tilde{\Omega}'$ where $\tilde{\Omega} = A(\Omega)$ and $\tilde{\Omega}' = h^{-1}A^{-1}(\Omega')$.

Furthermore, the following properties were established for doubling measures by Caffarelli [C3] and Jhaveri-Savin [JS].

Proposition 2.3. [JS, Section 3] *There exist constant $c > 0$ depends only on dimension, and $C \ge 1$ depending additionally on the doubling constant of $g(x)$ and $g'(y)$ such that*

(1) $B_{C^{-1}} \subset \nabla\tilde{u}(\tilde{S}) \subset B_C$

$$\implies C^{-1}h(S_h^c(u,x_0) - x_0)^\circ \subset \nabla(u - l)(S_h^c(u,x_0)) \subset Ch(S_h^c(u,x_0) - x_0)^\circ.$$

(2) $C^{-1}h^n \le |S_h^c(u,x_0)||\nabla u(S_h^c(u,x_0))| \le Ch^n$.

(3) $S_{ch}^c(u,x_0) \cap \overline{\Omega} \subset S_h(x_0) \subset S_{Ch}^c(u,x_0) \cap \overline{\Omega}$.

(4) $S_{C^{-1}h}(v,\nabla u(x_0)) \subset \nabla u(S_h(u,x_0)) \subset S_{Ch}(v,\nabla u(x_0))$.

In [C3, JS], it was also shown that if the measures are doubling, then the centered sections satisfy the following engulfing property.

Proposition 2.4. [JS, Lemma 3.3] *For any pair of constants $0 \le \underline{t} \le \overline{t} \le 1$, there exists a constant $0 < s \le 1$ such that*

$$S_{sh}^c(u,x') \subset \overline{t}S_h^c(u,x)$$

for all $x \in \overline{\Omega}$ and $x' \in \underline{t}S_h^c(u,x) \cap \overline{\Omega}$. Moreover, the constant s depend only on $\underline{t}, \overline{t}, n$ and the doubling constant of g and g'.

This engulfing property leads to the following $C^{1,\delta}$ estimate [C2, JS]. (see also [SaYu])

Proposition 2.5. *Given an optimal transport map* $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ *with* $0 \in \partial\Omega \cap \partial\Omega'$ *and* $u(0) = |\nabla u|(0) = 0$. *Assume that it satisfies*

(1) g *and* g' *has doubling constant bounded by* K.
(2) $S_1^c(u, 0) \subset B_K(0)$ *and* $S_1^c(v, 0) \subset B_K(0)$.

Then there exist $\delta > 0$ *depending only on* K *and* n *such that*

$$\|\bar{u}\|_{C^{1,\delta}(B_R)} + \|\bar{v}\|_{C^{1,\delta}(B_R)} \leq C$$

for C *depending only on* n, K, *and* R.

Proof. The proof follows from [SaYu, Proposition 2.5, Remark 2.1] using Proposition 2.4. □

3. Monotonicity formula

Definition 3.1. *Let* $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ *be an optimal transport map, where* $g \in C^{1,\alpha}(\overline{\Omega})$ *(resp.* $g' \in C^{1,\alpha}(\overline{\Omega'})$*). We say that* $g(x)$ *(resp.* $g'(y)$*) is* **weakly sub-homogeneous** *of degree* l, *with constants* (C, δ) *(resp. degree* l*) if there are constants* $C, \delta \geq 0$, *independent of* r, *such that*

$$\int_{D_r} x \cdot \nabla g(x)dx \leq (l + Cr^\delta) \int_{D_r} g(x)dx$$

and

$$\int_{D_r} y \cdot \nabla g'(y)dy \leq (k + Cr^\delta) \int_{D_r} g'(y)dy.$$

By convention, if $C = 0$ *we take* $\delta = 1$.

Proposition 3.2. *Let* $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ *be an optimal transport map with where* $(g(x), g'(y))$ *are positive, bounded uniformly away from zero,* $g \in C^{1,\alpha}_{loc}(\Omega) \cap C^\alpha(\overline{\Omega})$, $g' \in C^{1,\alpha}_{loc}(\Omega') \cap C^\alpha(\overline{\Omega'})$. *Assume in addition that* (g, g') *are weakly sub-homogeneous degree* (l, k) *with constants* (C, δ) *in the sense of Definition 3.1. Assume that* Ω, Ω' *are smooth and strictly convex,* $0 \in \overline{\Omega} \cap \overline{\Omega'}$. *Define the quantity*

$$\chi(r) = r^{-\frac{2(n+l)}{1+\frac{n+l}{n+k}}} \mu(D_r(u, 0)) = r^{-\frac{2(n+k)}{1+\frac{n+k}{n+l}}} \nu(D_r(v, 0)),$$

where

$$\mu(D_r(u, 0)) = \mu(\{x \cdot \nabla u(x) \leq r^2\}) = \int_{\{x \cdot \nabla u(x) \leq r^2\}} g(x)dx.$$

Then there is a constant $\beta = \beta(n, k, l)$ *such that* $e^{-\frac{\beta}{\delta}Cr^\delta}\chi(r)$ *is monotone non-increasing in* r.

Proof. Since Ω, Ω' are smooth, strictly convex and g, g' are positive, $C^{1,\alpha}$ regular in the interior and C^α up to the boundaries, we have that $u \in C^{3,\alpha}_{loc}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$ and $v \in C^{3,\alpha}_{loc}(\Omega') \cap C^{2,\alpha}(\overline{\Omega'})$ by Caffarelli's regularity theory [C1, C2, C3].

Let $f(x) = -\frac{1}{2}\log(g'(\nabla u(x))g(x))$ and define the Laplacian

$$\begin{aligned}
\Delta_f \phi &:= \frac{1}{\sqrt{\det D^2 u}\, e^{-f}} \partial_i(u^{ij}\sqrt{\det D^2 u}\, e^{-f}\partial_j\phi) = \frac{1}{g}\partial_i(u^{ij} g \partial_j \phi) \\
&= u^{ij}\phi_{ij} - u^{ik}u^{pj}u_{ikp}\phi_j + u^{ij}\frac{1}{g}g_i\phi_j.
\end{aligned}$$

The main observation is that we have the following formulas:

$$\begin{aligned}
\Delta_f u &= n + \sum_j \frac{1}{g'}\frac{\partial g'}{\partial y_j} u_j \\
\Delta_f (x \cdot \nabla u(x)) &= 2n + \sum_j \frac{1}{g'}\frac{\partial g'}{\partial y_j} u_j + \sum_j \frac{1}{g}\frac{\partial g}{\partial x_j} x_j
\end{aligned} \tag{3.1}$$

Indeed, by differentiating the optimal transport

$$\det D^2 u = \frac{g}{g'(\nabla u)}$$

with respect to x_p, we have

$$\frac{g}{g'(\nabla u)} u^{ik} u_{ikp} = \frac{1}{g'(\nabla u)} \frac{\partial g}{\partial x_p} - \frac{g}{(g'(\nabla u))^2} \frac{\partial g'}{\partial y_m} u_{mp},$$

i.e.

$$u^{ik} u_{ikp} = \frac{1}{g} g_p - \frac{1}{g'} \frac{\partial g'}{\partial y_m} u_{mp}.$$

So

$$\begin{aligned}
\Delta_f u &= u^{ij} u_{ij} - u^{ik} u^{pj} u_{ikp} u_j + u^{ij} \frac{1}{g} g_i u_j \\
&= n - u^{pj} u_j \left(\frac{1}{g} g_p - \frac{1}{g'} \frac{\partial g'}{\partial y_m} u_{mp} \right) + u^{ij} \frac{1}{g} g_i u_j \\
&= n - u^{pj} \frac{1}{g} g_p u_j + \frac{1}{g'} \frac{\partial g'}{\partial y_j} u_j + u^{ij} \frac{1}{g} g_i u_j \\
&= n + \sum_j \frac{1}{g'} \frac{\partial g'}{\partial y_j} u_j.
\end{aligned}$$

For each $1 \le m \le n$, we have

$$\begin{aligned}
\Delta_f (x \cdot \nabla u(x)) &= \sum_m \Delta_f (x_m u_m) = \sum_m x_m \Delta_f u_m + u_m \Delta_f x_m + 2 \\
&= 2n + \sum_m \Bigg[x_m \left(u^{ij} u_{mij} - u^{ik} u^{jl} u_{ikl} u_{mj} + u^{ij} \frac{1}{g} g_i u_{mj} \right) \\
&\qquad + u_m \left(-u^{ik} u^{jl} u_{ikl} \delta_{mj} + u^{ij} \frac{1}{g} g_i \delta_{mj} \right) \Bigg] \\
&= 2n + \sum_m \left[x_m \frac{1}{g} \frac{\partial g}{\partial x_m} + u_m \left(-u^{ik} u^{ml} u_{ikl} + u^{im} \frac{1}{g} g_i \right) \right] \\
&= 2n + \sum_m \left[x_m \frac{1}{g} \frac{\partial g}{\partial x_m} + u_m \left(-u^{ml} \left(\frac{1}{g} g_l - \frac{1}{g'} \frac{\partial g'}{\partial y_j} u_{jl} \right) + u^{im} \frac{1}{g} g_i \right) \right] \\
&= 2n + \sum_m \frac{1}{g'} \frac{\partial g'}{\partial y_j} u_m + \sum_m \frac{1}{g} \frac{\partial g}{\partial x_m} x_m.
\end{aligned}$$

Denote $\psi = x \cdot \nabla u(x)$ and consider the function $\phi = \gamma u + \psi$ for some $\gamma \in \mathbb{R}_{>0}$ to be determined. For $\kappa > 0$, by coarea formula we compute

$$\frac{d}{dr} \left(r^{-\kappa} \int_{\{\psi \le r\} \cap \Omega} g(x)\, dx \right) = -\kappa r^{-(\kappa+1)} \int_{\{\psi \le r\} \cap \Omega} g(x) dx + r^{-\kappa} \int_{\{\psi = r\} \cap \Omega} \frac{g(x)}{|\nabla \psi|} d\mathcal{H}^{n-1}(x)$$

By (3.1), we have

$$\begin{aligned}
\Delta_f \phi &= \gamma \Delta_f u + \Delta_f (x \cdot \nabla u(x)) \\
&= \gamma \left(n + \sum_j \frac{1}{g'} \frac{\partial g'}{\partial y_j} u_j \right) + 2n + \sum_j \frac{1}{g'} \frac{\partial g'}{\partial y_j} u_j + \sum_j \frac{1}{g} \frac{\partial g}{\partial x_j} x_j
\end{aligned}$$

$$= 2n + \gamma n + (1+\gamma)k + l + (1+\gamma)\left(\sum_j \frac{1}{g'}\frac{\partial g'}{\partial y_j}u_j - k\right) + \left(\sum_j \frac{1}{g}\frac{\partial g}{\partial x_j}x_j - l\right)$$
$$= \lambda + (1+\gamma)\left(\sum_j \frac{1}{g'}\frac{\partial g'}{\partial y_j}u_j - k\right) + \left(\sum_j \frac{1}{g}\frac{\partial g}{\partial x_j}x_j - l\right).$$

where $\lambda = \gamma(n+k) + 2n + k + l$. Then we have

$$\begin{aligned}\int_{\{\psi\le r\}\cap\Omega} g(x)dx &= \frac{1}{\lambda}\int_{\{\psi\le r\}\cap\Omega}(\Delta_f\phi)g(x)dx - \frac{(1+\gamma)}{\lambda}\int_{\{\psi\le r\}\cap\Omega}\left(\sum_j \frac{1}{g'}\frac{\partial g'}{\partial y_j}u_j - k\right) g(x)dx \\ &\quad - \frac{1}{\lambda}\int_{\{\psi\le r\}\cap\Omega}\left(\sum_j \frac{1}{g}\frac{\partial g}{\partial x_j}x_j - l\right) g(x)dx \\ &\ge \frac{1}{\lambda}\int_{\{\psi\le r\}\cap\Omega}(\Delta_f\phi)g(x)dx - C\left[\frac{(1+\gamma)}{\lambda} + \frac{1}{\lambda}\right] r^{\delta/2}\int_{\{\psi\le r\}\cap\Omega} g(x)dx \\ &= \frac{1}{\lambda}\int_{\{\psi\le r\}\cap\Omega}(\Delta_f\phi)\, g(x)dx - \frac{(2+\gamma)}{\lambda}Cr^{\delta/2}\int_{\{\psi\le r\}\cap\Omega} g(x)dx.\end{aligned} \tag{3.2}$$

where we use the fact that (g, g') are weakly sub-homogeneous of degree (l, k) with constants $C(,\delta)$ and the optimal transport equation. Precisely, we have

$$\int_{\{\psi\le r\}\cap\Omega}\left(\sum_j \frac{1}{g}\frac{\partial g}{\partial x_j}x_j - l\right) g(x)dx = \int_{\{\psi\le r\}\cap\Omega} x\cdot\nabla g\,\mathrm{d}x - l\int_{\{\psi\le r\}\cap\Omega} g(x)\,\mathrm{d}x \le Cr^{\delta/2}\int_{\{\psi\le r\}\cap\Omega} g(x)dx,$$

$$\begin{aligned}\int_{\{\psi\le r\}\cap\Omega}\left(\sum_j \frac{1}{g'}\frac{\partial g'}{\partial y_j}u_j - k\right) g(x)dx &\overset{y=\nabla u(x)}{=} \int_{\{\psi\le r\}^*\cap\Omega'}\left(\sum_j \frac{1}{g'}\frac{\partial g'}{\partial y_j}y_j - k\right) g'(y)dy \\ &\le Cr^{\delta/2}\int_{\{\psi\le r\}^*\cap\Omega'} g'(y)dy = Cr^{\delta/2}\int_{\{\psi\le r\}\cap\Omega} g(x)dx.\end{aligned}$$

It remains to analyze the first term. Let $\nu_{\partial\Omega}$ denote the outward pointing normal vector to $\partial\Omega$. We can now integrate-by-parts to get

$$\frac{1}{\lambda}\int_{\{\psi\le r\}\cap\Omega}(\Delta_f\phi)\, g(x)dx = \frac{1}{\lambda}\int_{\Omega\cap\{\psi=r\}}\langle\nabla\phi,\nabla\psi\rangle_u\frac{g(x)}{|\nabla\psi|}\,d\mathcal{H}^{n-1}(x) + \frac{1}{\lambda}\int_{\partial\Omega\cap\{\psi\le r\}}\langle\nabla\phi,\nu_{\partial\Omega}\rangle_u g(x)\,d\mathcal{H}^{n-1}(x) \tag{3.3}$$

Now we compute

$$\begin{aligned}\langle\nabla\phi,\nabla\psi\rangle_u &= \gamma\langle\nabla u,\nabla\psi\rangle_u + \langle\nabla\psi,\nabla\psi\rangle_u \\ &= \gamma u^{ij}u_i(u_j + \sum_m x_m u_{mj}) + u^{ij}\left((u_i + \sum_p x_p u_{pi})((u_j + \sum_m x_m u_{mj})\right) \\ &= (1+\gamma)u^{ij}u_iu_j + \gamma\sum_m x_m u_m + 2\sum_m x_m u_m + \sum_{p,m} x_p x_m u_{mp} \\ &= (1+\gamma)|\nabla u|_u^2 + \|x\|_u^2 + (2+\gamma)\psi.\end{aligned}$$

We remark the notion. For vectors V, W, we denote by $V\cdot W$ the euclidean inner product, and use $|V|$ for the euclidean norm of V. We use

$$\langle V, W\rangle_u := V^iW^ju^{ij}, \qquad |V|_u^2 = \langle V, V\rangle_u, \qquad (V, W)_u := V^iW^ju_{ij}, \qquad \|V\|_u^2 := (V, V)_u.$$

Note that $\|\nabla_g u\|_u^2 = u_k u^{ki} u_l u^{lj} u_{ij} = u^{kl} u_k u_l = |\nabla u|_u^2$, $(\nabla_g u, x)_u = u_k u^{ki} x_j u_{ij} = x_j u_j = x \cdot \nabla u = \psi$. We have

$$\begin{aligned}\langle \nabla\phi, \nabla\psi\rangle_u &= (\sqrt{1+\gamma}\nabla_g u, \sqrt{1+\gamma}\nabla_g u)_u + (x,x)_u + \left[(\sqrt{1+\gamma}+1)^2 - 2\sqrt{1+\gamma}\right](\nabla_g u, x)_u \\ &= \|(\sqrt{1+\gamma})\nabla_g u - x\|_u^2 + (1+\sqrt{1+\gamma})^2\psi.\end{aligned}$$

In total, we have

$$\begin{aligned}\frac{d}{dr}\left(r^{-\kappa}\int_{\Omega\cap\{\psi\le r\}} g(x)dx\right) &= -\kappa r^{-(\kappa+1)}\int_{\Omega\cap\{\psi\le r\}} g(x)dx + r^{-\kappa}\int_{\Omega\cap\{\psi=r\}}\frac{g(x)}{|\nabla\psi|}d\mathcal{H}^{n-1}(x) \\ &\le -\frac{\kappa}{\lambda}r^{-(\kappa+1)}\int_{\Omega\cap\{\psi=r\}}\|(\sqrt{1+\gamma})\nabla_g u - x\|_u^2\frac{g(x)}{|\nabla\psi|}\,d\mathcal{H}^{n-1}(x) \\ &\quad -\frac{\kappa}{\lambda}r^{-(\kappa+1)}\int_{\Omega\cap\{\psi=r\}}\left((1+\sqrt{1+\gamma})^2 - \frac{\lambda}{\kappa}\right)\psi\frac{g(x)}{|\nabla\psi|}\,d\mathcal{H}^{n-1}(x) \\ &\quad -\frac{\kappa}{\lambda}r^{-(\kappa+1)}\int_{\partial\Omega\cap\{\psi\le r\}}\langle\nabla\phi, \nu_{\partial\Omega}\rangle_u g(x)\,d\mathcal{H}^{n-1}(x) \\ &\quad +\frac{\kappa(2+\gamma)}{\lambda}r^{-(\kappa+1)}Cr^{\delta/2}\int_{\{\psi\le r\}} g(x)dx\end{aligned}$$

Choose $\kappa = \kappa(\gamma)$ so that $\frac{\lambda}{\kappa} = (1+\sqrt{1+\gamma})^2$; that is

$$\begin{aligned}\kappa(\gamma) &= \frac{\gamma(n+k)+2n+k+l}{(1+\sqrt{1+\gamma})^2} = \frac{\left[(\sqrt{1+\gamma})^2-1\right](n+k)+2n+k+l}{(\sqrt{1+\gamma}+1)^2} \\ &= \frac{\sqrt{1+\gamma}-1}{\sqrt{1+\gamma}+1}(n+k) + \frac{2n+k+l}{(\sqrt{1+\gamma}+1)^2} \\ &= (2n+k+l)\left(\frac{1}{\sqrt{1+\gamma}+1}\right)^2 - 2(n+k)\left(\frac{1}{\sqrt{1+\gamma}+1}\right) + n+k.\end{aligned}$$

and now choose $\gamma = \gamma_* > 0$ so that $\kappa_* = \kappa(\gamma_*)$ is minimized. We have

$$\sqrt{1+\gamma_*} = \frac{n+l}{n+k} \qquad \kappa_* = \frac{(n+k)(n+l)}{(n+k)+(n+l)} \tag{3.4}$$

For these choices we obtain

(3.5)

$$\begin{aligned}\frac{d}{dr}\left(r^{-\kappa_*}\int_{\Omega\cap\{\psi\le r\}} g dx\right) &\le -(1+\sqrt{1+\gamma_*})^{-2}r^{-(\kappa_*+1)}\left(\int_{\Omega\cap\{\psi=r\}}\|(\sqrt{1+\gamma_*})\nabla_g u - x\|_u^2\frac{g(x)}{|\nabla\psi|}\,\mathrm{d}\mathcal{H}^{n-1}(x)\right) \\ &\quad -(1+\sqrt{1+\gamma_*})^{-2}r^{-(\kappa_*+1)}\int_{\partial\Omega\cap\{\psi\le r\}}\langle\nabla\phi, \nu_{\partial\Omega}\rangle_u g(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &\quad +(1+\sqrt{1+\gamma_*})^{-2}(2+\gamma_*)Cr^{\frac{\delta}{2}-1}\left(r^{-\kappa_*}\int_{\Omega\cap\{\psi\le r\}} g(x)\,\mathrm{d}x\right)\end{aligned}$$

We now analyze the boundary term. We have

$$\langle\nabla\phi, \nu_{\partial\Omega}\rangle_u = u^{ij}(\gamma_* u_i + u_i + x_l u_{li})(\nu_{\partial\Omega})_j = (1+\gamma_*)u_i u^{ij}(\nu_{\partial\Omega})_j + x_j(\nu_{\partial\Omega})_j$$

Since Ω is convex, and $0\in\overline{\Omega}$ we have

$$x\cdot\nu_{\partial\Omega}\ge 0$$

with equality if and only if $0\in\partial\Omega$ and Ω is conical near 0. Similarly, we note that $u^{ij}(\nu_{\partial\Omega})_j(x)$ is normal to $\partial\Omega'$ at $y=\nabla u(x)$. Thus, since $0\in\overline{\Omega'}$, the convexity of Ω' yields

$$u_i u^{ij}(\nu_{\partial\Omega})_j(x)\ge 0$$

with equality if and only if $0 \in \partial\Omega'$ and Ω' is conical near 0. Thus, we obtain

$$\frac{d}{dr}\left(r^{-\kappa_*}\int_{\Omega\cap\{\psi\le r\}} gdx\right) \le (1+\sqrt{1+\gamma_*})^{-2}(2+\gamma_*)Cr^{\frac{\delta}{2}-1}\left(r^{-\kappa_*}\int_{\Omega\cap\{\psi\le r\}} gdx\right).$$

Multiplying $\exp\left(-\frac{2\tilde{C}}{\delta}r^{\delta/2}\right)$ where $\tilde{C} = (1+\sqrt{1+\gamma_*})^{-2}(2+\gamma_*)C$, we obtain

$$\frac{d}{dr}\left(e^{-\frac{2\tilde{C}}{\delta}r^{\delta/2}}r^{-\kappa_*}\int_{\Omega\cap\{\psi\le r\}} gdx\right) \le 0.$$

So we can take $\beta = 2(1+\sqrt{1+\gamma_*})^{-2}(2+\gamma_*)$ where γ_* as (3.4), to obtain the desired result. □

Remark 3.3. *When Ω, Ω' are C^2 and strictly convex, $g > 0$ on $\overline{\Omega}$, and $g' > 0$ on $\overline{\Omega'}$, and (g, g') are homogeneous of degree (l, k), the proof of Proposition 3.2 yields the desired monotonicity formula, but the equality case can never occur due to the strict convexity of the boundaries.*

Remark 3.4. *In the proof of Proposition 3.2, the $C^{2,\alpha}$ boundary regularity of u, v is only needed to justify the integration by parts formula (3.3). In particular, we observe that $C^{2,\alpha}$ boundary regularity is only needed near points of $\partial(\Omega\cap\{\psi\le r\})$.*

Theorem 3.5. *Let $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal transport map with*

(i) $g \in C^\alpha(\overline{\Omega})$, $\Omega \subset \{g > 0\}$, and g is homogeneous of degree l.

(ii) $g' \in C^\alpha(\overline{\Omega'})$, $\Omega' \subset \{g' > 0\}$, and g' is homogeneous of degree k.

Assume $0 \in \overline{\Omega}\cap\overline{\Omega'}$ and $0 = u(0) = |\nabla u|(0)$. Then the quantity

$$\chi(r) := r^{-\frac{2(n+l)}{1+\frac{n+l}{n+k}}}\mu(D_r(u,0))$$

is monotone non-increasing in r, where recall

$$\mu(D_r(u,0)) = \int_{\{\psi(x)\le r^2\}} g(x)dx$$

for $\psi(x) := x\cdot\nabla u(x)$. More precisely, for any $r_2 < r_1$ we have

$$\chi(r_1)-\chi(r_2) \le -2(1+\frac{n+l}{n+k})^2\int_{r_2}^{r_1} s^{-(1+\frac{2(n+l)(n+k)}{2n+l+k})}\int_{\Omega\cap\{\psi=s\}}\|(\frac{n+l}{n+k})\nabla_g u - x\|_u^2\frac{g(x)}{|\nabla\psi|}\,d\mathcal{H}^{n-1}(x)ds$$

where $\|V\|_u^2 = V^iV^ju_{ij}$. In particular, $\chi(r)$ is constant on $[0, r_0]$ if and only if Ω and Ω' are both conical about the origin and u is homogeneous of degree $1+\frac{n+l}{n+k}$ in $D_{r_0}(u, 0)$.

We remark for the particular case. $\chi(r)$ is constant if and only if

$$\|(\frac{n+l}{n+k})\nabla_g u - x\|_u = 0 \text{ in } \Omega, \qquad \langle\nabla\phi, \nu_{\partial\Omega}\rangle_u = 0 \text{ on } \partial\Omega.$$

Equivalently, Ω and Ω' are both conical about the origin and

$$x_i = (\frac{n+l}{n+k})u^{ki}u_k \Leftrightarrow x_iu_{il} = (\frac{n+l}{n+k})u^{ki}u_ku_{il} = (\frac{n+l}{n+k})u_l \Leftrightarrow (x_iu_i)_l - u_l = \left[(\frac{n+l}{n+k})u\right]_l,$$

Combing the fact that $u(0) = 0$, we have

$$x\cdot\nabla u = \left[1+(\frac{n+l}{n+k})\right]u,$$

which means u is homogeneous of degree $1+\frac{n+l}{n+k}$.

Proof. (Approximation argument) Suppose that (u, v) satisfy the assumptions of the theorem. We may assume that $(g(x), g'(y))$ are globally defined, non-negative homogeneous functions, and by assumption $\Omega \subset \{g > 0\}$ and $\Omega' \subset \{g' > 0\}$. We assume that $g, g' \in C^{\alpha}_{loc}(\mathbb{R}^n)$. Choose a sequence of positive, smooth functions $(g_\varepsilon(x), g'_\varepsilon(y))$, homogeneous of degree (l, k) converging in C^α on compact sets to $(g(x), g'(y))$. Define

$$\widehat{g}_\varepsilon(x) = g_\varepsilon(x) + \varepsilon$$
$$\widehat{g}'_\varepsilon(y) = g'_\varepsilon(y) + \varepsilon$$

Fix $R \geq 1$ and suppose that u is defined on $\Omega \cap B_{2R}$ (if Ω is compact, choose $R \gg 1$ so that $\Omega \subset B_{R/2}$). For a set A, let $B_\delta(A)$ denote the δ neighborhood of A. Fix a $1 \gg \delta > 0$ and define a compact set K by

$$K = \overline{B_{3\delta/2}(\nabla u(\Omega \cap B_{R/2}))}$$

where δ is chosen small so that

$$K \cap B_\delta(\nabla u(\Omega \cap \partial B_R)) = \emptyset.$$

If Ω is compact, then $K = \overline{B_\delta(\Omega')}$ for $\delta \ll 1 \ll R$. We choose a sequence of sets $\Upsilon_\varepsilon, \Upsilon'_\varepsilon$ with the following properties:

(i) $\Upsilon_\varepsilon, \Upsilon'_\varepsilon$ are smoothly bounded domains, with $0 \in \overline{\Upsilon_\varepsilon} \cap \overline{\Upsilon_{\varepsilon'}}$.
(ii) Υ_ε is uniformly convex, and as $\varepsilon \to 0$, Υ_ε converges to $\Omega \cap B_R$ in the Hausdorff sense.
(iii) $\Upsilon'_\varepsilon \supset \nabla u(\Omega \cap B_R)$, and as $\varepsilon \to 0$, Υ'_ε converges to $\nabla u(\Omega \cap B_R)$ in the Hausdorff sense.
(iv) For ε sufficiently small, the portion of $\partial \Upsilon'_\varepsilon$ intersecting K is uniformly convex.
(v) The mass balancing condition holds:

$$\int_{\Upsilon_\varepsilon} \widehat{g}_\varepsilon(x)dx = \int_{\Upsilon'_\varepsilon} \widehat{g}'_\varepsilon(y)dy$$

Such a sequence can be constructed as follows. If Ω, Ω' are compact, then we can take $\Omega_\varepsilon \supset \Omega$ and $\Omega'_\varepsilon \supset \Omega'$ to be smooth, uniformly convex domains such that $d_H(\Omega, \Omega_\varepsilon) = d_H(\Omega', \Omega'_\varepsilon) \leq \varepsilon$. Take $\Upsilon_\varepsilon = \Omega_\varepsilon$, and choose Υ'_ε to be a dilation of Ω'_ε such that the mass balancing condition (v) holds. Since the mass balancing condition holds for $\varepsilon = 0$ it follows that the dilation factors converge to 1, and hence the Υ'_ε converge to Ω' in the Hausdorff sense, as desired.

Now suppose that Ω, Ω' are non-compact convex sets. In this case the construction is complicated slightly by the fact that $\nabla u(\Omega \cap B_R)$ may not be convex. First, choose $R \gg 1$ so that $\nabla v(0) \in \overline{\Omega \cap B_{R/2}}$ and take $\Omega_{\varepsilon,R}$ to be a sequence of smoothly bounded, uniformly convex sets, with $\Omega_{\varepsilon,R} \supset \Omega \cap B_R$ and such that $d_H(\Omega_{\varepsilon,R}, \Omega \cap B_R) \leq \varepsilon$ in the Hausdorff sense. To construct Υ'_ε we define

$$A := \mathrm{ConvexHull}(\nabla u(\Omega \cap B_R))$$

and let $A_{\varepsilon'} \supset A$ be a sequence of smoothly bounded, uniformly convex sets converging to A in the Hausdorff sense. For ε' small, we choose $\Upsilon'_{\varepsilon'} \supset \nabla u(\Omega \cap B_R)$ to be a smoothly bounded domain agreeing with $A_{\varepsilon'}$ in K, and converging to $\nabla u(\Omega \cap B_R)$ as $\varepsilon' \to 0$. The construction is depicted in Figure 1.

Given ε, choose $\varepsilon'(\varepsilon)$ small so that $d_H(\Upsilon'_{\varepsilon'}, \nabla u(\Omega \cap B_R)) = d_H(\Omega_{\varepsilon,R}, \Omega \cap B_R)$. Finally, take Υ_ε to be a small dilation of $\Omega_{\varepsilon,R}$ in order to enforce the mass balancing.

Given $r_0 > 0$ after, possibly increasing R we may assume that

$$\{x \cdot \nabla u(x) < 2r_0\} \Subset \Omega \cap B_{R/2}$$

Let v_ε be the convex function solving the optimal transport problem

$$\begin{aligned} \widehat{g}_\varepsilon(\nabla v_\varepsilon) \det D^2 v_\varepsilon &= \widehat{g}'_\varepsilon(y), \quad \text{in } \Upsilon'_\varepsilon \\ \nabla v_\varepsilon(\Upsilon'_\varepsilon) &= \Upsilon_\varepsilon \end{aligned} \tag{3.6}$$

The regularity theory for optimal transport yields the following

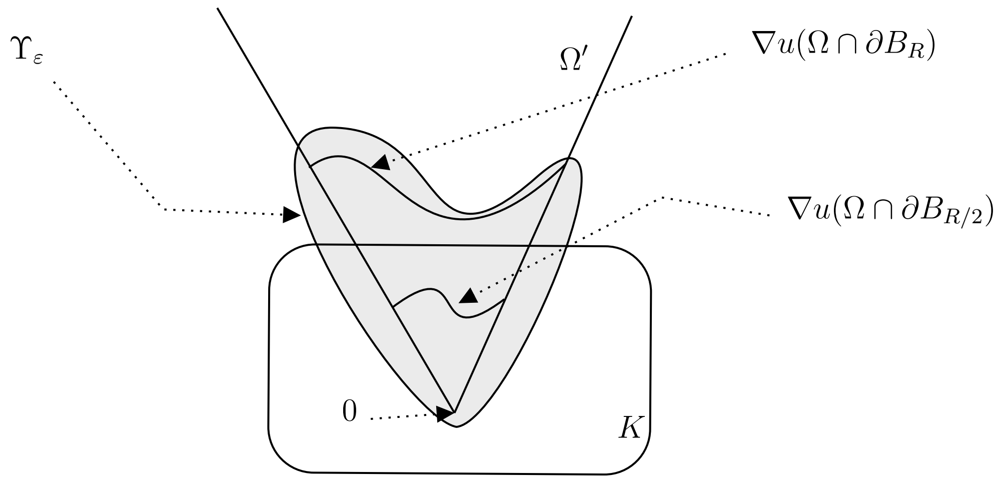


FIGURE 1. A diagrammatic representation of the construction of Υ'_ε, denoted by the gray region, in the case that Ω, Ω' are not compact

- By [C1, C2, C3, JS], v_ε are uniformly bounded in $C^{1,\alpha}(\overline{\Upsilon'_\varepsilon})$. Furthermore, $v_\varepsilon \to v$ locally uniformly in $\nabla u(\Omega \cap B_R)$.
- By Caffarelli's regularity theory [C2, C3], and properties $(i), (ii), (iv)$ of the approximation, for every fixed ε, v_ε are $C^{3,\alpha}_{loc}(\Upsilon'_\varepsilon)$, and $C^{2,\alpha}(\overline{\Upsilon'_\varepsilon} \cap K)$. Furthermore, since $\widehat{g}_\varepsilon$, and $\widehat{g}'_\varepsilon$ are uniformly bounded below, and C^α, v_ε converge to v in $C^{2,\alpha}_{loc}(\Omega \cap B_{R/2})$
- u_ε are $C^{3,\alpha}_{loc}(\Upsilon_\varepsilon)$. By $(i), (ii), (iv)$ and by [C2, C3] , $u_\varepsilon \in C^{2,\alpha}(\nabla v_\varepsilon(K \cap \overline{\Upsilon'_\varepsilon}))$. Since v_ε converges to v in $C^{1,\alpha}(\overline{\Upsilon'_\varepsilon})$, for ε sufficiently small we have
$$\{y \cdot \nabla v_\varepsilon(y) < r_0\} \subset B_\delta(\{y \cdot \nabla v(y) < 2r_0\}) \subset K \cap \overline{\Upsilon'_\varepsilon}$$
and hence $u_\varepsilon \in C^{2,\alpha}(\overline{\{x \cdot \nabla u_\varepsilon(x) < r_0\}})$.
- As $\varepsilon \to 0$, u_ε converges in $C^{2,\alpha}_{loc}(\Omega \cap B_R)$ to u, and v_ε converges in $C^{2,\alpha}_{loc}(\nabla u(\Omega \cap B_{R/2}))$ to v.

We can now apply the result of Proposition 3.2 with $(C, \delta) = (0, 1)$, keeping in mind Remark 3.4. We conclude that, for any $r_2 < r_1 < r_0$
$$\begin{aligned} &r_1^{-\kappa_*} \int_{\Upsilon_\varepsilon \cap \{\psi_\varepsilon < r_1\}} \widehat{g}_\varepsilon(x)dx - r_2^{-\kappa_*} \int_{\Upsilon_\varepsilon \cap \{\psi_\varepsilon < r_1\}} \widehat{g}_\varepsilon(x)dx \\ &\le -(1 + \sqrt{1+\gamma_*})^2 \int_{r_2}^{r_1} s^{-(\kappa_*+1)} \int_{\Upsilon_\varepsilon \cap \{\psi_\varepsilon = s\}} |(\sqrt{1+\gamma_*})\nabla u_\varepsilon - x|^2_{u_\varepsilon} \frac{\widehat{g}_\varepsilon(x)}{|\nabla \psi_\varepsilon|}\, ds\, d\mathcal{H}^{n-1}(x) \end{aligned}$$
where κ_*, γ_* are given by (3.4), and $\psi_\varepsilon = x \cdot \nabla u_\varepsilon(x)$. The result now follows from Fatou's lemma. □

We can also prove an effective version of the monotonicity formula for Hölder continuous densities.

The key is the following lemma, which establishes the weak sub-homogeneity $(0, \varepsilon)$ of non-degenerate Hölder continuous densities at small (but fixed) scales.

Lemma 3.6. *Given an optimal transport map $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ where the densities $(g(x), g'(y))$ are smooth and satisfy*
$$C^{-1} \le g(x), g'(y) \le C$$
and
$$[g]_{\alpha,\overline{\Omega}} + [g']_{\alpha,\overline{\Omega'}} \le C$$
for some $\alpha \in (0,1)$ and $C > 1$. Then there exist $r_0 > 0$, $B > 0$, and $\varepsilon > 0$ depending only on n, α, C such that we have
$$\int_{D_r} (x \cdot \nabla g)dx \le Br^\varepsilon \int_{D_r} g(x)dx.$$

for all $r < r_0$.

Proof. We write the integral in radial coordinates

$$\int_{D_r} (x \cdot \nabla g) dx = \int_{\{x \cdot \nabla u \leq r^2\}} (x \cdot \nabla g) \, \mathrm{d}x = c_n \int_{t \in S^{n-1}} \int_0^{\phi(t)} (s(\nabla_t g)(st)) s^{n-1} \, \mathrm{d}s \mathrm{d}\sigma(t) \tag{3.7}$$

where $\phi(t)$ is chosen so that $(\phi(t)t) \cdot \nabla u(\phi(t)t) = r^2$. We can integrate by parts along radial rays to get rid of the derivative on g

$$\int_0^\phi g'(s) s^n ds = g(\phi)\phi^n - n \int_0^\phi g(s) s^{n-1} ds = n \int_0^\phi (g(\phi) - g(s)) s^{n-1} ds \leq C \phi^\alpha \int_0^\phi s^{n-1} ds.$$

Putting this into equation (3.7), we get

$$\int_{D_r} (x \cdot \nabla g) dx \leq C(\mathrm{diam}(D_r))^\alpha \int_{D_r} dx.$$

The constant C controls the doubling constant of (g, g'), hence by the uniform convexity estimate (Proposition 2.5), we have the fact:

For any $x \in \Omega$, denote $y = Du(x) = Du(Dv(y))$. Proposition 2.5 implies

$$|x| = |Dv(y)| \leq C|y|^\alpha = C|Du(x)|^\alpha \;\; \Rightarrow \;\; \frac{|u(x)|}{|x|} \approx |Du(x)| \geq C^{-1}|x|^{\frac{1}{\alpha}} \Rightarrow \;\; |u(x)| \geq C|x|^{1+\frac{1}{\alpha}}.$$

If

$$r^2 \geq u(x) \geq C|x|^{1+\frac{1}{\alpha}},$$

then

$$|x| \leq C r^{\overbrace{\frac{2}{1+\frac{1}{\alpha}}}^{=:\delta \in (0,1)}}.$$

So we have

$$D_r(u, 0) \subset S_{r^2}(u, 0) \subset B_{Cr^\delta}(0)$$

for some $\delta > 0$.

This implies $|g(x) - g(0)| \leq C r^{\delta\alpha}$, therefore for $r < \frac{1}{(2C^2)^{\frac{1}{\delta\alpha}}}$, we have $g(x) \geq g(0) - \frac{C^{-1}}{2} \geq \frac{C^{-1}}{2}$, and $\mathrm{diam}(D_r) \leq C r^\delta$. Hence we have

$$\int_{D_r} (x \cdot \nabla g) dx \leq B r^{\delta\alpha} \int_{D_r} g(x) dx.$$

□

Proposition 3.7. *Given an optimal transport map $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ where the densities $(g(x), g'(y))$ satisfy*

$$C^{-1} \leq g(x), g'(y) \leq C$$

and

$$[g]_{\alpha, \overline{\Omega}} + [g']_{\alpha, \overline{\Omega'}} \leq C$$

for some $\alpha \in (0,1)$ and $C > 1$. Then there exist $A > 1$, $\varepsilon_0 > 0$, and $r_0 > 0$ depending only on n, α, and C such that for any $x_0 \in \overline{\Omega}$, the quantity

$$\chi(x_0, r) = e^{-Ar^{\varepsilon_0}} r^{-n} \int_{D_r(u, x_0)} g(x) dx$$

is monotone non-increasing in r for $r \in (0, r_0]$.

Proof of Proposition 3.7. If (g, g') are smooth and (Ω, Ω') have smooth boundaries, then this follows from combining Lemma 3.6 and Proposition 3.2. For the general case, we can approximate (Ω, Ω') by smooth domains, and (g, g') by smooth densities, and apply the same approximation argument as in the proof of Theorem 3.5. □

4. Homogeneity of Blow-ups

In this section, we use the monotonicity formula to show that the blow-ups of optimal transport maps are always homogeneous. We will treat two cases, the first when the densities are non-degenerate and Hölder continuous, and the second case when the densities are homogeneous.

First let us define what a blow-up is.

Definition 4.1. *$((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal transport map with $0 \in \overline{\Omega} \cap \overline{\Omega'}$ such that $u(0) = |\nabla u|(0) = 0$. We call an optimal transport map*

$$((\Omega_\infty, g_\infty(x)dx), (\Omega'_\infty, g'_\infty(y)dy), u_\infty, v_\infty)$$

a **blow-up** *of $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ at $0 \in \overline{\Omega}$ along scales $h_i \to 0$, if there is $R > 1$ and a sequence of positive symmetric matrices A_{h_i} satisfying*

$$B_{R^{-1}}(0) \subset A_{h_i}(S^c_{h_i}(u, 0)) \subset B_R(0)$$

and a sequence of positive constants $c_i > 0$, such that the sequence of rescaled optimal transport maps $((\Omega_i, g_i(x)dx), (\Omega'_i, g'_i(y)dy), u_i, v_i)$ given by

$$\Omega_i := A_{h_i}(\Omega), \qquad \Omega'_i := h_i^{-1} A_{h_i}^{-1}(\Omega'),$$

$$g_i(x) := c_i g(A_{h_i}^{-1} x), \qquad g'_i(y) := c_i h_i^n (\det A_{h_i})^2 g'(h_i A_{h_i} y),$$

$$u_i(x) := \frac{u(A_{h_i}^{-1} x)}{h_i}, \qquad v_i(y) := \frac{v(h_i A_{h_i} y)}{h_i},$$

converges to $((\Omega_\infty, g_\infty(x)dx), (\Omega'_\infty, g'_\infty(y)dy), u_\infty, v_\infty)$ in the following sense

(1) $(\Omega_i, \Omega'_i) \to (\Omega_\infty, \Omega'_\infty)$ in the locally Hausdorff sense.
(2) $(g_i(x)dx, g'_i(y)dy) \to (g_\infty(x)dx, g'_\infty(y)dy)$ in the weak sense of measures.
(3) The minimal convex extensions $(\bar{u}_i, \bar{v}_i)$ converges to the minimal convex extensions $(\bar{u}_\infty, \bar{v}_\infty)$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$.

4.1. With Hölder continuous density.

In this section, we prove the **existence** and **homogeneity** of blow-ups when $(g(x), g'(y))$ are non-degenerate and Hölder continuous.

We first prove a uniform density estimate when $(g(x), g'(y))$ are non-degenerate and Hölder continuous on general convex domains. This estimate improves upon the previously known uniform density estimates ([C3, Section 3], [CLW, Section 2]), which required additional regularity assumptions on the boundary of $\partial\Omega$.

Lemma 4.2 (Uniform density). *Given an optimal transport map $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ where the densities $(g(x), g'(y))$ satisfy*

$$C^{-1} \le g(x), g'(y) \le C$$

and

$$[g]_{\alpha,\overline{\Omega}} + [g']_{\alpha,\overline{\Omega'}} \le C$$

for some $\alpha \in (0, 1)$ and $C > 1$. Then for any $x_0 \in \overline{\Omega}$ and $0 < h \le 1$, we have

$$\frac{|S_h(u, x_0)|}{h^{\frac{n}{2}}} \ge c|S_1(u, 0)|$$

for $c > 0$ depending only on n, α and C.

Proof. By Proposition 3.7, there exist A, h_0 and ε_0 depending on C such that for any $0 < h < h_0$, we have

$$h^{-\frac{n}{2}} \mu(D_{h^{\frac{1}{2}}}(u, 0)) \ge \chi(0, h^{1/2}) \ge \chi(0, h_0^{1/2}) = e^{-Ah_0^{\frac{\varepsilon_0}{2}}} h_0^{-\frac{n}{2}} \mu(D_{h_0^{\frac{1}{2}}}(u, 0)).$$

Hence

$$
\begin{aligned}
|S_h(u,0)| &\geq |D_{h^{\frac{1}{2}}}(u,0)| \geq C^{-1}\int_{D_{h^{1/2}}} g(x)dx = c\mu(D_{h^{\frac{1}{2}}}(u,0)) \\
&\geq ce^{-Ah_0^{\frac{\varepsilon_0}{2}}} h_0^{-\frac{n}{2}}\mu(D_{h_0^{\frac{1}{2}}}(u,0))h^{\frac{n}{2}} \geq ch^{\frac{n}{2}}|D_{h_0^{\frac{1}{2}}}(u,0)| \\
&\geq ch^{\frac{n}{2}}|S_{h_0}(u,0)| \geq ch^{\frac{n}{2}}|S_1(u,0)|.
\end{aligned}
$$

□

Theorem 4.3 (Homogeneity of blow-ups)**.** *Let $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal transport map with $0 \in \overline{\Omega} \cap \overline{\Omega'}$, and suppose $g(x), g'(y)$ are positive and Hölder continuous on $\overline{\Omega}, \overline{\Omega'}$ respectively. Then for any sequence of scales $h_i \to 0$, there is a subsequence along which a blow-up exists and is of the form $((\mathtt{C}, cdx), (\mathtt{C}', c'dy), u_\infty, v_\infty)$ for some constants $c, c' > 0$. Moreover, $(\mathtt{C}, \mathtt{C}')$ are cones about the origin and (u_∞, v_∞) are homogeneous of degree 2.*

Proof. By Lemma 4.2, we know that,

$$
|S_h(u,0)| \geq ch^{\frac{n}{2}}, \qquad |S_h(v,0)| \geq ch^{\frac{n}{2}}.
$$

Since the $(g(x), g'(y))$ are both doubling measures, Proposition 2.3(2)(3)(4) also implies

$$
|S_h(u,0)||S_h(v,0)| \leq Ch^n,
$$

and hence we have

$$
|S_h^c(u,0)| \sim |S_h^c(v,0)| \sim |S_h(u,0)| \sim |S_h(v,0)| \sim h^{\frac{n}{2}}.
$$

Hence without loss of generality, we can assume the normalizing matrices are given by $A_h = h^{-\frac{1}{2}}M_h$ with $\det M_h = 1$. It follows that for any $R > 1$, the rescaled domains $\Omega_i \cap B_R(0) = A_{h_i}(\Omega) \cap B_R(0)$ have a uniform lower bound $(h_0^{-n/2}|\Omega|)$ on its volume, and therefore we can extract a subsequence for which the convex domains (Ω_i, Ω_i') converges to convex domains $(\mathtt{C}, \mathtt{C}')$ locally in the Hausdorff sense, and moreover $(\mathtt{C}, \mathtt{C}')$ have non-empty interior. The rescaled potentials

$$
u_i(x) := \frac{u(h_i^{\frac{1}{2}}M_{h_i}^{-1}x)}{h_i}, \qquad v_i(y) = \frac{v(h_i^{\frac{1}{2}}M_{h_i}y)}{h_i},
$$

satisfy

$$
(\nabla u_i)_\sharp(g(h_i^{\frac{1}{2}}M_{h_i}^{-1}x)dx) = g'(h_i^{\frac{1}{2}}M_{h_i}y)dy.
$$

Moreover by Proposition 2.5, we have $\|M_h\| + \|M_h^{-1}\| \leq h^{-\frac{1}{2}+\delta}$ for some $\delta > 0$, and hence by the continuity of $g(x)$ and $g'(y)$, we know that

$$
g_i(x) := g(h_i^{\frac{1}{2}}M_{h_i}^{-1}x) \to g(0)
$$

and

$$
g_i'(y) := g'(h_i^{\frac{1}{2}}M_{h_i}y) \to g'(0)
$$

as $h_i \to 0$. Therefore after passing to a subsequence, we set $0 < c := g(0)$ and $0 < c' := g'(0)$, then it follows by Proposition 2.5 that a subsequence of $(\bar{u}_i, \bar{v}_i)$ converges in $C^{1,\alpha}_{loc}$ to a limiting function $(\bar{u}_\infty, \bar{v}_\infty)$, which gives an optimal transport map $((\mathtt{C}, cdx), (\mathtt{C}', c'dy), u_\infty, v_\infty)$.

Moreover, we have

$$
\lim_{i\to\infty}\left(h_i^{-\frac{1}{2}}r\right)^{-n}\int_{D_{h_i^{\frac{1}{2}}r}(u,0)} g(x)dx \overset{z=h_i^{-1/2}x}{=} \lim_{i\to\infty} r^{-n}\int_{D_r(u_i,0)} g_i(x)dx = g(0)r^{-n}\int_{D_r(u_\infty,0)} dx
$$

hence the volume ratio $\phi_\infty(r) = r^{-n}|D_r(u_\infty, 0)|$ of this blow-up is constant, i.e.

$$\chi(0,r) = r^{-\frac{2(n+0)}{1+\frac{n+0}{n+0}}}\mu(D_r(u_\infty,0)) = r^{-n}\int_{D_r(u_\infty,0)} g(0)\,\mathrm{d}x$$
$$= g(0)r^{-n}|D_r(u_\infty,0)| = c \quad \text{since } |D_r(u_{\infty,0})| \sim r^n$$
$$(= \lim_{r\to 0} g(0)^{-1}e^{-Ar^{\varepsilon_0}}r^{-n}\mu(D_r(u,0)) > 0.)$$

It follows from rigidity case of the monotonicity formula (Theorem 3.5) that $(\mathtt{C}, \mathtt{C}')$ are cones and (u_∞, v_∞) are homogeneous of degree $2 = 1 + \frac{n+0}{n+0}$. □

Remark 4.4. *We note that if $((\mathtt{C}, cdx), (\mathtt{C}', c'dy), u_\infty, v_\infty)$ is a blow-up of the optimal transport map $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ at $0 \in \partial\Omega$, it is not necessarily the case that the cones $\mathtt{C}$ and $\mathtt{C}'$ are affine equivalent to the tangent cones of Ω and Ω' at 0. This is because the sequence of affine transformations A_h may degenerate and change the shape of the convex sets in the limit. However, this is true if the sections of u are "round" (see Definition 4.5).*

Definition 4.5. *Let $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal transport map with homogeneous densities $(g(x), g'(y))$ of degree (l, k). Suppose $0 \in \partial\Omega\cap\partial\Omega'$ and $0 = u(0) = |\nabla u|(0)$. We say that the sections of (u, v) at 0 are* **round** *if there is a constant $C > 1$ and $h_0 > 0$ such that for all $0 < h < h_0$, we have*

$$B_{C^{-1}h^{\frac{1}{1+\frac{n+l}{n+k}}}}(0) \subset S^c_h(u,0) \subset B_{Ch^{\frac{1}{1+\frac{n+l}{n+k}}}}(0)$$

and

$$B_{C^{-1}h^{\frac{1}{1+\frac{n+k}{n+l}}}}(0) \subset S^c_h(v,0) \subset B_{Ch^{\frac{1}{1+\frac{n+k}{n+l}}}}(0).$$

4.2. **With homogeneous density.** In this section, we consider the case where $g(x)$ and $g'(y)$ are homogeneous of degree l and k respectively.

Theorem 4.6 (Homogeneity of blow-ups)**.** *Let $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal transport map with C^α densities $(g(x), g'(y))$ which are homogeneous of degree (l, k) for $k, l \geq 0$. Suppose that the sections $S^c_h(u,0)$ satisfy*

$$B_{C^{-1}h^{\frac{1}{1+\frac{n+l}{n+k}}}}(0) \subset S^c_h(u,0) \subset B_{Ch^{\frac{1}{1+\frac{n+l}{n+k}}}}(0).$$

Then for any sequence of scales $h_i \to 0$, there exists a subsequence along which a blow-up exists, and is of the form $((\mathtt{C}, g(x)dx), (\mathtt{C}', g'(y)dy), u_\infty, v_\infty)$. Moreover $(\mathtt{C}, \mathtt{C}')$ are the tangent cones of (Ω, Ω') at 0, and (u_∞, v_∞) are homogeneous of degree $1 + \frac{n+l}{n+k}$ and $1 + \frac{n+k}{n+l}$ respectively.

Proof. By the assumption on the roundeness of the section we can take $A_h = h^{-\frac{1}{1+\frac{n+l}{n+k}}} Id$ when normalizing $S^c_h(u,0)$. The rescaled functions are given by

$$u_i(x) = \frac{u(h_i^{\frac{1}{1+\frac{n+l}{n+k}}}x)}{h_i}, \qquad v_i(y) = \frac{v(h_i^{\frac{1}{1+\frac{n+k}{n+l}}}y)}{h_i},$$

and it follows that $((h_i^{-\frac{1}{1+\frac{n+l}{n+k}}}\Omega, g(x)dx), (h_i^{-\frac{1}{1+\frac{n+k}{n+l}}}\Omega', g'(y)dy), u_i, v_i)$ are a sequence of optimal transport maps. By our rescaling, we have $(h_i^{-\frac{1}{1+\frac{n+l}{n+k}}}\Omega, h_i^{-\frac{1}{1+\frac{n+k}{n+l}}}\Omega') \to (\mathtt{C}, \mathtt{C}')$ where $(\mathtt{C}, \mathtt{C}')$ is the tangent cone of (Ω, Ω') at 0. By Proposition 2.5, after taking a subsequence, $(\bar{u}_i, \bar{v}_i)$ will converge to some limiting optimal tranpsort map $(\bar{u}_\infty, \bar{v}_\infty)$ in $C^{1,\alpha}_{loc}$.

Moreover, the equality in the monotonicity formula is achieved for the limit, which implies that (u_∞, v_∞) are homogeneous of degree $1 + \frac{n+l}{n+k}$ and $1 + \frac{n+k}{n+l}$ respectively. □

Remark 4.7. *Notice that the existence of a homogeneous density breaks the affine invariance of the problem, which is why we need the additional assumption on the shape of the sections to extract a homogeneous blow-up.*

5. Global regularity for Hölder continuous densities

5.1. **$C^{1,1-\varepsilon}$ and $W^{2,p}$ regularity.** In this section, we prove that an optimal transport map between convex domains equipped with non-degenerate Hölder continuous densities is globally $C^{1,1-\varepsilon}$ for any $\varepsilon > 0$ and $W^{2,p}$ for any $p > 1$. This result is optimal, as $C^{1,1}$-regularity does not always hold without any additional regularity assumption on the boundary. The case when $n = 2$ and $g = g' = 1$ was previously proved by Savin-Yu [SaYu].

Proposition 5.1. *Suppose $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ is an optimal transport map with $0 \in \overline{\Omega}$, and assume $g(x), g'(y)$ are positive and Hölder continuous in $\overline{\Omega}, \overline{\Omega'}$ respectively. Then for any $\varepsilon > 0$, there exist $h_0 > 0$ such that for all $0 < h < h_0$, we have*

$$(1-\varepsilon)\left(\frac{1}{2}\right)^{\frac{1}{2}} S_h^c(u,0) \subseteq S_{\frac{h}{2}}^c(u,0) \subseteq (1+\varepsilon)\left(\frac{1}{2}\right)^{\frac{1}{2}} S_h^c(u,0).$$

Proof. Since the two inclusion follows from exactly the same argument, we will only prove the second inclusion. Suppose it is not true, then there exist sequence $h_i \to 0$ such that

$$S_{\frac{h_i}{2}}^c(u,0) \nsubseteq (1+\varepsilon)(\frac{1}{2})^{\frac{1}{2}} S_{h_i}^c(u,0).$$

It follows that

$$A_{h_i}(S_{\frac{h_i}{2}}^c(u,0)) \nsubseteq (1+\varepsilon)(\frac{1}{2})^{\frac{1}{2}} A_{h_i}(S_{h_i}^c(u,0)).$$

But by Theorem 4.3, we know that after taking a subsequence we can extract a homogeneous blow-up limit $((\mathtt{C}, cdx), (\mathtt{C}', c'dy), u_\infty, v_\infty)$, hence we have the convergence of centered sections

$$\begin{aligned} A_{h_i}(S_{\frac{h_i}{2}}^c(u,0)) &= \{A_{h_i}x \in \Omega : u(x) \le u(0) + p\cdot x + \frac{h_i}{2}\} \\ &= \{y \in \Omega : u(A_{h_i}^{-1}y) \le u(0) + p\cdot A_{h_i}^{-1}y + \frac{h_i}{2}\} \\ &= \{y \in \Omega : u_i(y) \le u_i(0) + \tilde{p}\cdot y + \frac{1}{2}\} \to S_{\frac{1}{2}}^c(u_\infty, 0). \end{aligned}$$

and $A_{h_i}(S_{h_i}^c(u,0)) \to S_1^c(u_\infty,0)$ similarly, which implies

$$S_{\frac{1}{2}}^c(u_\infty,0) \nsubseteq (1+\frac{\varepsilon}{2})(\frac{1}{2})^{\frac{1}{2}} S_1^c(u_\infty,0).$$

But this is a contradiction because u_∞ is homogeneous of degree 2 and hence

$$\begin{aligned} S_{\frac{1}{2}}^c(u_\infty,0) &= \left\{x \in \Omega : u_\infty(x) < u_\infty(0) + p\cdot x + \frac{1}{2}\right\} \\ &= \{x \in \Omega : 2u_\infty(x) < 2u_\infty(0) + 2p\cdot x + 1\} \\ &= \left\{x \in \Omega : u_\infty(\sqrt{2}x) < u_\infty(0) + \tilde{p}\cdot\sqrt{2}x + 1\right\} = \frac{1}{\sqrt{2}} S_1^c(u_\infty,0). \end{aligned}$$

□

Corollary 5.2. *Given any optimal transport map $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ with $0 \in \overline{\Omega}$ and $g(x), g'(y)$ are positive and Hölder continuous in $\overline{\Omega}, \overline{\Omega'}$. Then for any $\varepsilon > 0$, there exist $h_0 > 0$ depending on n, ε, and $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ such that for all $0 < h < h_0$, we have*

$$h^{\frac{1}{2}+\varepsilon} S_1^c(u,0) \subset S_h^c(u,0) \subset h^{\frac{1}{2}-\varepsilon} S_1^c(u,0).$$

Proof. This follows by iteratively applying Proposition 5.1 to u for ε small enough. We outline the proof here.

For convenience, we denote $S_h := S_h^c(u,0)$. We only consider $h = 2^{-m}$. By Proposition 5.1, we have

$$S_h \subset \frac{1+\varepsilon}{\sqrt{2}} S_{2^{-m+1}} \subset \cdots \subset \left(\frac{1+\varepsilon}{\sqrt{2}}\right)^m S_1 = (1+\varepsilon)^m h^{1/2} S_1,$$
$$S_h \supset \frac{1-\varepsilon}{\sqrt{2}} S_{2^{-m+1}} \supset \cdots \supset \left(\frac{1-\varepsilon}{\sqrt{2}}\right)^m S_1 = (1-\varepsilon)^m h^{1/2} S_1.$$

Now we take

$$\delta = \min\left\{2^{\varepsilon} - 1, 1 - 2^{-\varepsilon}\right\},$$

and repalce ε by δ to obtain

$$h^{1/2+\varepsilon} S_1 \subset 2^{-m\varepsilon} h^{1/2} S_1 \subset (1-\delta)^m h^{1/2} S_1 \subset S_h \subset (1+\varepsilon)^m h^{1/2} S_1 \subset 2^{m\varepsilon} h^{1/2} S_1 = h^{1/2-\varepsilon} S_1.$$

□

Now we make this bound effective.

Proposition 5.3. *Let* $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ *be an optimal transport map where the densities* $(g(x), g'(y))$ *satisfy*

$$C^{-1} \le g(x), g'(y) \le C$$

and

$$[g]_{\alpha,\overline{\Omega}} + [g']_{\alpha,\overline{\Omega'}} \le C$$

for some $\alpha \in (0,1)$ *and* $C > 1$. *Suppose* $0 \in \overline{\Omega} \cap \overline{\Omega'}$, *and* $0 = u(0) = |\nabla u|(0)$, *and there exist* $R > 1$ *such that*

$$B_{R^{-1}}(0) \subset S_1^c(u,0) \subset B_R(0). \tag{5.1}$$

Then for any $\varepsilon > 0$, *there exist* $0 < h_0 < \frac{1}{2}$ *depending only on* n, α, C, ε, *and* R *such that we have*

$$h^{\frac{1}{2}+\varepsilon} S_1^c(u,0) \subset S_h^c(u,0) \subset h^{\frac{1}{2}-\varepsilon} S_1^c(u,0)$$

for some $h \in [h_0, \frac{1}{2}]$.

Proof. Suppose it is not true, then there exists a sequence $((\Omega_i, g_i(x)dx), (\Omega_i', g_i'(y)dy), u_i, v_i)$ satisfying the stated bounds such that given any $h \in (0, \frac{1}{2}]$, for i sufficiently large, we have either

$$h^{\frac{1}{2}+\varepsilon} S_1^c(u_i,0) \not\subset S_h^c(u_i,0) \text{ or } S_h^c(u_i,0) \not\subset h^{\frac{1}{2}-\varepsilon} S_1^c(u_i,0).$$

However, this sequence of optimal transport maps is compact. Indeed by Lemma 4.2, (Ω_i, Ω_i') converges up to a subsequence in the locally Hausdorff sense to some $(\Omega_\infty, \Omega_\infty')$. By the Hölder bounds, the densities $(g_i(x), g_i'(y))$ converges in $C^{\alpha'}$ for any $0 < \alpha' < \alpha$ to $(g_\infty(x), g_\infty'(y))$, which is positive and Hölder continuous.

Moreover, by Proposition 2.5 the potentials $(\overline{u}_i, \overline{v}_i)$ converge in $C^{1,\gamma}_{loc}$ to a limiting potential $(\overline{u}_\infty, \overline{v}_\infty)$, which gives an optimal transport map $((\Omega_\infty, g_\infty(x)dx), (\Omega_\infty', g_\infty'(y)dy), u_\infty, v_\infty)$. It follows from Corollary 5.2 that for some $0 < \varepsilon' < \varepsilon$, there exist some $0 < h < \frac{1}{2}$ for which

$$h^{\frac{1}{2}+\varepsilon'} S_1^c(u_\infty,0) \subset S_h^c(u_\infty,0) \subset h^{\frac{1}{2}-\varepsilon'} S_1^c(u_\infty,0).$$

But this contradicts our assumption since we have

$$S_h^c(u_i,0) \to S_h^c(u_\infty,0) \qquad S_1^c(u_i,0) \to S_1^c(u_\infty,0).$$

□

Proposition 5.4. *Let $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal transport map where the densities $(g(x), g'(y))$ satisfy*

$$C^{-1} \leq g(x), g'(y) \leq C$$

and

$$[g]_{\alpha,\overline{\Omega}} + [g']_{\alpha,\overline{\Omega'}} \leq C$$

for some $\alpha \in (0,1)$ and $C > 1$. Suppose $0 \in \overline{\Omega} \cap \overline{\Omega'}$, and $0 = u(0) = |\nabla u|(0)$, and there exist $R > 1$ such that

$$B_{R^{-1}}(0) \subset S_1^c(u, 0) \subset B_R(0). \tag{5.2}$$

Then for any $\varepsilon > 0$, there exist $M > 1$ depending only on n, α, C, and R such that

$$M^{-1} h^{\frac{1}{2}+\varepsilon} S_1^c(u, 0) \subset S_h^c(u, 0) \subset M h^{\frac{1}{2}-\varepsilon} S_1^c(u, 0)$$

for all $h \leq 1$.

Proof. By the uniform density estimate (Lemma 4.2), for any sequence $h_i \to 0$ and a sequence of normalization matrix $A_{h_i} = h_i^{\frac{1}{2}} M_{h_i}$ with $\det M_{h_i} = 1$, the rescaled optimal transport maps $((\Omega_i, g_i(x)dx), (\Omega_i', g_i'(y)dy), u_i, v_i)$ with $g_i(x) = g(h_i^{\frac{1}{2}} M_{h_i}^{-1} x)$, $g_i'(y) = g'(h_i^{\frac{1}{2}} M_{h_i} y)$ satisfy the hypothesis of Proposition 5.3 uniformly for some new constants α, C, R. Therefore, we can iterate Proposition 5.3 to get the result. □

From this, we immediately obtain the global $C^{1,1-\varepsilon}$ regularity of optimal transport maps.

Theorem 5.5 (Global $C^{1,1-\varepsilon}$ regularity)**.** *Let $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal transport map where the densities $(g(x), g'(y))$ satisfy*

$$C^{-1} \leq g(x), g'(y) \leq C$$

and

$$[g]_{\alpha,\overline{\Omega}} + [g']_{\alpha,\overline{\Omega'}} \leq C$$

for some $\alpha \in (0,1)$ and $C > 1$. Then for any $\varepsilon > 0$, we have $u \in C^{1,1-\varepsilon}(\overline{\Omega})$ and $v \in C^{1,1-\varepsilon}(\overline{\Omega'})$ and moreover, we have the estimate

$$\|\nabla u\|_{C^{1-\varepsilon}(\overline{\Omega})} + \|\nabla v\|_{C^{1-\varepsilon}(\overline{\Omega'})} \leq M$$

where M depends only on n, α, C, ε, and the inner and outer radius of Ω and Ω'.

Proof. Since the inner and outer radius is controlled, let us denote them by r and R, so we have $\Omega, \Omega' \subset B_R(0)$ and $B_r(x_0) \subset \Omega$. Then from the outer radius bound for Ω', we know that $|\nabla u| \leq R$. Moreover, the inner radius bound implies that for some $h_0 > R^2 \geq \operatorname{osc}_{\overline{\Omega}} u$, we have $S_{h_0}(u, x) \supset B_r(x_0)$ for all $x \in \overline{\Omega}$, which implies

$$|S_{h_0}(u, x)| \geq c r^n > 0.$$

Therefore by Proposition 5.4, we have $\|\nabla u\|_{C^\alpha(\overline{\Omega})} \leq C$. The same argument applies to v. □

The argument of [CLW, Theorem 1.2], [SaYu, Theorem 1.1] can be applied to also give $W^{2,p}$-estimates.

Theorem 5.6 (Global $W^{2,p}$ regularity)**.** *Let $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal transport map where the densities $(g(x), g'(y))$ satisfy*

$$C^{-1} \leq g(x), g'(y) \leq C$$

and

$$[g]_{\alpha,\overline{\Omega}} + [g']_{\alpha,\overline{\Omega'}} \leq C$$

for some $\alpha \in (0,1)$ *and* $C > 1$. *Then for any* $p > 1$, $u \in W^{2,p}(\overline{\Omega})$ *and* $v \in W^{2,p}(\overline{\Omega'})$, *and moreover, we have the estimate*

$$\|\nabla u\|_{W^{1,p}(\overline{\Omega})} + \|\nabla v\|_{W^{1,p}(\overline{\Omega'})} \le M$$

where M *depends only on* n, α, C, p, *and the inner and outer radius of* Ω *and* Ω'.

Proof. This follows from the argument of [SaYu, Theorem 1.1] using Proposition 5.4. □

5.2. **$C^{2,\alpha}$-regularity for domains with $C^{1,\alpha}$-boundary.** In this section, we show that if Ω, Ω' have $C^{1,\beta}$-boundary, and $g(x), g'(y)$ are C^{α}, then in fact the solution to optimal transport problem is $C^{2,\min(\alpha,\beta)}$. This is sharp, and improves previous results of Caffarelli [C3] and Chen-Liu-Wang [CLW], which required the domains to have $C^{1,1}$-boundary.

First, we prove a lemma which says that if the boundary is $C^{1,\alpha}$, then any blow-up has to live on a half-space.

Lemma 5.7. *Let* $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ *be an optimal transport map where the densities* $g(x), g'(y)$ *are positive and Hölder continuous in* $\overline{\Omega}, \overline{\Omega'}$ *respectively. Suppose* $0 \in \partial\Omega \cap \partial\Omega'$ *with* $u(0) = |\nabla u|(0) = 0$, $\Omega \subset \{x_n \ge 0\}$ *and locally near* 0,

$$\overline{\Omega} \cap B_1 = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n \ge \phi(x')\} \cap B_1$$

where $\phi : B_1^{n-1} \to \mathbb{R}$ *is convex and for all* $x' \in B_1^{n-1}$, *we have*

$$0 \le \phi(x') \le C|x'|^{1+\alpha}.$$

Given $h_i \to 0$, *let* A_{h_i} *be the sequence of normalizing matrices for* $S^c_{h_i}(u, 0)$, *then for any* $R > 1$, *one has*

$$\lim_{h_i \to 0} \frac{|A_{h_i}\Omega \cap B_R|}{|B_R|} \to \frac{1}{2}.$$

In particular, the domains $A_{h_i}\Omega$ *and* $A_{h_i}\{x_n > 0\}$ *both converge to some half-space* H.

Proof. Without loss of generality, we can assume $A_h = h^{-\frac{1}{2}} M_h$ with $\det M_h = 1$ and from Proposition 5.4 we also have $\|M_h\| \ll h^{-\varepsilon}$ for any $\varepsilon > 0$. Therefore, we have

$$A_h\Omega = M_h\{x_n \ge h^{-\frac{1}{2}}\phi(h^{\frac{1}{2}}x')\}$$

and we have

$$|A_h\Omega \cap B_R| = |\{x_n \ge h^{-\frac{1}{2}}\phi(h^{\frac{1}{2}}x')\} \cap M_h^{-1}B_R| \ge |\{x_n \ge Ch^{\frac{\alpha}{2}}|x'|^{1+\alpha}\} \cap M_h^{-1}B_R|.$$

We can bound $|A_h\Omega \cap B_R|$ as follows

$$\begin{aligned}
\frac{1}{2}|B_R| &\ge |A_h\Omega \cap B_R| \\
&\ge |\{x_n \ge Ch^{\frac{\alpha}{2}}|x'|^{1+\alpha}\} \cap M_h^{-1}B_R| \\
&= |\{x_n \ge 0\} \cap M_h^{-1}B_R| - |\{0 \le x_n \le Ch^{\frac{\alpha}{2}}|x'|^{1+\alpha}\} \cap M_h^{-1}B_R| \\
&\ge \frac{1}{2}|B_R| - |\{0 \le x_n \le Ch^{\frac{\alpha}{2}}|x'|^{1+\alpha}\} \cap B_{h^{-\varepsilon}R}| \\
&\ge \frac{1}{2}|B_R| - Ch^{\frac{\alpha}{2}} \int_{B^{n-1}_{h^{-\varepsilon}R}} |x'|^{1+\alpha} dx' \\
&\ge \frac{1}{2}|B_R| - C(n, \alpha, R)h^{\frac{\alpha}{2} - \varepsilon(n+\alpha)}
\end{aligned}$$

where from third line to fourth line, we used $\|M_h\| \ll h^{-\varepsilon}$. It follows that if ε is sufficiently small, then as $h \to 0$, we have

$$|A_h\Omega \cap B_R| \to \frac{1}{2}|B_R|.$$

It's clear the $A_{h_i}\{x_n \geq 0\}$ must converge to some half-space H, and by the volume bound, the $A_{h_i}\Omega$ is then a subset of H, but has the same volume, therefore it must also be all of H as well. □

Lemma 5.8 (Obliqueness)**.** *Let $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal transport map where the densities $g(x), g'(y)$ are positive and Hölder continuous in $\overline{\Omega}, \overline{\Omega'}$ respectively. Suppose $u(0) = |\nabla u|(0) = 0$, and the domains Ω and Ω' are $C^{1,\alpha}$ at 0 for some $\alpha > 0$. If $l_\Omega, l_{\Omega'}$ are the defining functions for the supporting hyperplane of Ω and Ω' at 0, then we have*

$$l_\Omega \cdot l_{\Omega'} > 0.$$

Proof. Suppose this is not true, then $l_\Omega \cdot l_{\Omega'} = 0$. Then we consider a blow up of $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ at 0 along some subsequence $h_i \to 0$. From Lemma 5.7, we know that

$$A_{h_i}\Omega \to H$$
$$h_i A_{h_i}^{-1}\Omega' \to H'$$

where H, H' are half-spaces defined by defining functions l_H and $l_{H'}$ so that $H = \{x \in \mathbb{R}^n : l_H(x) \geq 0\}$ and $H' = \{y \in \mathbb{R}^n : l_{H'}(y) \geq 0\}$, and

$$\lim_{h_i \to 0} \frac{l_\Omega \circ A_{h_i}^{-1}}{|l_\Omega \circ A_{h_i}^{-1}|} = \frac{l_H}{|l_H|}$$

and

$$\lim_{h_i \to 0} \frac{l_{\Omega'} \circ A_{h_i}}{|l_{\Omega'} \circ A_{h_i}|} = \frac{l_{H'}}{|l_{H'}|}.$$

Moreover, since the blow-up is a homogeneous optimal transport map from (H, cdx) to $(H', c'dy)$, we know that

$$l_H \cdot l_{H'} > 0.$$

However, we also have

$$\frac{l_H \cdot l_{H'}}{|l_H||l_{H'}|} = \lim_{h_i \to 0} \frac{l_\Omega \circ A_{h_i}^{-1} \cdot l_{\Omega'} \circ A_{h_i}}{|l_\Omega \circ A_{h_i}^{-1}||l_{\Omega'} \circ A_{h_i}|} = 0$$

which is a contradiction. □

Once the obliqueness is established, the arguments of Chen-Liu-Wang [CLW, Section 6] can be applied to give $C^{2,\alpha}$-estimates.

Theorem 5.9. *Suppose $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ is an optimal transport map where the densities $(g(x), g'(y))$ satisfy*

$$C^{-1} \leq g(x), g'(y) \leq C$$

and

$$[g]_{\alpha,\overline{\Omega}} + [g']_{\alpha,\overline{\Omega'}} \leq C$$

for some $\alpha \in (0,1)$ and $C > 1$. Suppose in addition Ω, Ω' have $C^{1,\beta}$-boundary for some $\beta \in (0,1)$. Then $u \in C^{2,\gamma}(\overline{\Omega})$ and $v \in C^{2,\gamma}(\overline{\Omega'})$ for $\gamma = \min(\alpha, \beta)$ and

$$\|\nabla u\|_{C^{1,\gamma}(\overline{\Omega})} + \|\nabla v\|_{C^{1,\gamma}(\overline{\Omega'})} \leq M$$

for M that depends only on n, α, β, C, and the domains Ω, Ω'.

Proof. Once we have Lemma 5.8, we can follow the proof from [CLW, Section 6] with minor modifications. We'll explain the main modification: since $\partial\Omega, \partial\Omega'$ are only $C^{1,\beta}$, Lemma 6.2 of [CLW] only holds under the additional assumption $\delta < \frac{\beta}{2}$. Looking through the proof of [CLW, Lemma 6.2], we see that if we only have $C^{1,\beta}$-boundary, then we only have $a_h \leq h^{\frac{1+\beta}{2} - \varepsilon}$ (instead of $a_h \leq h^{1-\varepsilon}$) and

$$0 \leq D_n u \leq C_1 h^{\frac{1+\beta}{2} - \varepsilon}.$$

Therefore, for the lower barrier, we must use

$$\breve{w} := (1+h^\delta)^{\frac{1}{n}} w - (1+h^\delta)^{\frac{1}{n}} h + h + C_1(x_n - Ch^{\frac{1}{2}-\varepsilon}) h^{\frac{1+\beta}{2}-\varepsilon},$$

and the rest of the proof of [CLW, Lemma 6.2] holds assuming $\delta < \frac{\beta}{2}$. The rest of the arguments follow exactly as in [CLW, Section 6]. ☐

6. Roundness of sections

In this section we are interested in the studying the shapes of sections, which can be regarded as a weak form of $C^{1,1}$ regularity. We make the following definition.

Definition 6.1. *Let $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal transport map with homogeneous densities $(g(x), g'(y))$ of degree (l,k). Suppose $0 \in \partial\Omega \cap \partial\Omega'$ and $0 = u(0) = |\nabla u|(0)$. We say that the sections of (u,v) at 0 are* **round** *if there is a constant $C > 1$ and $h_0 > 0$ such that for all $0 < h < h_0$, we have*

$$B_{C^{-1}h^{\frac{1}{1+\frac{n+l}{n+k}}}}(0) \subset S_h^c(u,0) \subset B_{Ch^{\frac{1}{1+\frac{n+l}{n+k}}}}(0)$$

and

$$B_{C^{-1}h^{\frac{1}{1+\frac{n+k}{n+l}}}}(0) \subset S_h^c(v,0) \subset B_{Ch^{\frac{1}{1+\frac{n+k}{n+l}}}}(0).$$

If the sections of u at 0 are round, and $((\mathtt{C}, d\mu), (\mathtt{C}', d\nu), u_\infty, v_\infty)$ is a blow-up of u about 0, then it is clear that $(\mathtt{C}, \mathtt{C}')$ is affine equivalent to the tangent cone of (Ω, Ω') about 0. Thus, the roundness of sections is related to the existence of homogeneous optimal transport maps between the tangent cones of (Ω, Ω'). Using this idea, we first prove a negative result saying that roundness does not hold if homogeneous optimal transport maps between tangent cones do not exist.

Theorem 6.2. *Let $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal tranport map with homogeneous densities $(g(x), g'(y))$ of degree (l,k), such that $0 \in \partial\Omega \cap \partial\Omega'$ and $u(0) = |\nabla u|(0) = 0$. Let $(\mathtt{C}, \mathtt{C}')$ be the tangent cones of (Ω, Ω') at 0. If there is no homogenous optimal transport map from $(\mathtt{C}, g(x)dx)$ to $(\mathtt{C}', g'(y)dy)$, then the centered sections of (u,v) are not round at 0.*

Proof. Suppose that the centered sections are round, then $A_h = h^{-\frac{1}{1+\frac{n+l}{n+k}}} Id$ is a family of normalization matrices for $S_h^c(u,0)$, and by Theorem 4.6 the blow-up at 0 exists and must be homogeneous optimal transport map from $(\mathtt{C}, g(x)dx)$ to $(\mathtt{C}', g'(y)dy)$. But this contradicts our hypothesis that no such homogeneous map exist. ☐

A very natural question is then the following.

Question 6.3. *Suppose there exists a homogenous optimal transport map between $\mathtt{C}$ and $\mathtt{C}'$. Are the sections of u necessarily round at 0?*

We will address this question in subsequent sections.

6.1. **Strongly oblique case.**

Let $(\mathtt{C}, \mathtt{C}')$ be the tangent cone of (Ω, Ω') at 0 respectively, in this section, we prove a theorem that guarantees the roundness of sections under a *strong obliqueness* condition on the geometry of $(\mathtt{C}, \mathtt{C}')$.

Theorem 6.4. *Let $((\Omega, g(x)dx), (\Omega', g'(y)dy), u, v)$ be an optimal tranport map with homogeneous densities $(g(x), g'(y))$ of degree (l,k), such that $0 \in \partial\Omega \cap \partial\Omega'$ and $u(0) = |\nabla u|(0) = 0$. Let $(\mathtt{C}, \mathtt{C}')$ be the tangent cones of (Ω, Ω') at 0. If $\overline{\mathtt{C}'} \setminus \{0\} \subset int(\mathtt{C}^\circ)$, then the centered sections of (u,v) are round at 0. In particular, all blow-ups are affine equivalent to a homogenous optimal transport map between $\mathtt{C}$ and $\mathtt{C}'$.*

Proof. First we show the (non-centered) sections $S_h(u,0)$ and $S_h(v,0)$ are round. More precisely, we claim there exist $C>1$ depending on the cones $(\mathtt{C},\mathtt{C}')$ such that for all $h<1$, there exist $r(h),r'(h)>0$ such that

$$B_{C^{-1}r(h)}\cap\Omega\subset S_h(u,0)\subset B_{Cr(h)}\cap\Omega$$

and

$$B_{C^{-1}r'(h)}\cap\Omega'\subset S_h(v,0)\subset B_{Cr'(h)}\cap\Omega'.$$

To see this, we first use the geometric condition $\overline{\mathtt{C}'}\setminus\{0\}\subset\mathrm{int}(\mathtt{C}^\circ)$, i.e. for any $0\neq y\in\overline{\mathtt{C}'}$, $x\cdot y>0$ for any $x\in\mathtt{C}$, then the map

$$y\mapsto\inf_{x\in\mathtt{C},|x|=1}x\cdot y$$

is continuous and positive. Since $\overline{\mathtt{C}'}$ is compact, there exists $\delta>0$ such that

$$\inf_{y\in\overline{\mathtt{C}'},|y|=1}\left(\inf_{x\in\mathtt{C},|x|=1}x\cdot y\right)\geq\delta,$$

which implies there exist $\delta>0$ such that for all $x\in\Omega$ and $y\in\Omega'$,

$$x\cdot y\geq\delta|x||y|.$$

Now fix x_0 so that $u(x_0)=h$, then by the convexity of u, we have

$$u(x)\geq u(x_0)+\nabla u(x_0)\cdot(x-x_0)\geq h+\delta|\nabla u(x_0)||x|-|\nabla u(x_0)||x_0|$$

which implies that

$$S_h(u,0)\subset\{\delta|\nabla u(x_0)||x|-|\nabla u(x_0)||x_0|\leq 0\}\cap\Omega=B_{\delta^{-1}|x_0|}(0)\cap\Omega.$$

From this we see that if x_0,x_1 are two points in $\partial S_h(u,0)\cap\Omega$, then $|x_1|\leq\delta^{-1}|x_0|$, and the claim follows.

Next we show that the centered sections $S_h^c(u,0)$ and $S_h^c(v,0)$ are round as well. We note that $B_{C^{-1}r(h)}\cap\Omega\subset S_h(u,0)\subset B_{Cr(h)}\cap\Omega$ implies that $|S_h(u,0)|\sim r(h)^n$ and $|S_h(v,0)|\sim r'(h)^n$, and hence for all h sufficiently small, Proposition 2.3 gives

$$r(h)^nr'(h)^n\sim|S_h(u,0)||S_h(v,0)|\leq|S^c_{Ch}(u,0)||\nabla u(S^c_{Ch}(u,0))|\sim h^n$$

which implies

$$r'(h)r(h)\lesssim h.$$

Moreover, we also have

$$r(h)^{n+l}\lesssim\int_{B_{C^{-1}r(h)}}g(x)dx\leq\int_{S_h(u,0)}g(x)dx\leq\int_{B_{Cr(h)}}g(x)dx\lesssim r(h)^{n+l}$$

and

$$r'(h)^{n+k}\lesssim\int_{B_{C^{-1}r'(h)}}g'(y)dy\leq\int_{S_h(v,0)}g'(y)dy\leq\int_{B_{Cr'(h)}}g'(y)dy\lesssim r'(h)^{n+k},$$

hence by the monotonicity formula (Theorem 3.5), we have

$$r(h)^{n+l}\sim\int_{S_h(u,0)}g(x)dx\geq\int_{D_{h^{\frac{1}{2}}}(u,0)}g(x)dx\gtrsim h^{\frac{n+l}{1+\frac{n+l}{n+k}}}$$

and

$$r'(h)^{n+k}\sim\int_{S_h(v,0)}g'(y)dy\geq\int_{D_{h^{\frac{1}{2}}}(v,0)}g'(y)dy\gtrsim h^{\frac{n+k}{1+\frac{n+k}{n+l}}},$$

which implies

$$r(h)r'(h)\gtrsim h^{\frac{1}{1+\frac{n+l}{n+k}}+\frac{1}{1+\frac{n+k}{n+l}}}=h.$$

Hence we have $r(h) \sim h^{\frac{1}{1+\frac{n+l}{n+k}}}$ and $r'(h) \sim h^{\frac{1}{1+\frac{n+k}{n+l}}}$. This implies $|S_h(u,0)||S_h(v,0)| \sim h^n$, which implies by Proposition 2.3 that there is a uniform density bound

$$\frac{|S_h^c(u,0)\cap\Omega|}{|S_h^c(u,0)|}, \frac{|S_h^c(v,0)\cap\Omega'|}{|S_h^c(v,0)|} \geq \delta > 0.$$

If we let $l_1(h), \ldots, l_n(h)$ denote the lengths of the principle axis of the John ellipsoid of $S_h^c(u,0)$, then since $B_{cr(h)}(x') \subset S_{C^{-1}h}(u,0) \subset S_h^c(u,0)$, we have that

$$l_i(h) \gtrsim r(h) \text{ for all } i = 1, 2, \ldots, n.$$

Moreover the uniform density bound implies that

$$\prod_{i=1}^{n} l_i(h) \sim |S_h^c(u,0)| \sim |S_h(u,0)| \sim r(h)^n.$$

Altogether, we see that $l_i(h) \sim r(h)$ for all $i = 1, \ldots, n$, which implies that the centered sections of u are round. The same argument applied to v implies that the centered sections of v are round as well. $\square$

6.2. **2D domains with Lebesgue measure.** In this section, we focus on the case when $n = 2$. For simplicity, we also focus on the situation $g = g' = 1$. In particular, we'll give a complete answer to Question 6.3 when (Ω, Ω') are planar polytopes.

In this case, we have the following observation.

Proposition 6.5. *The only quadratic solutions to* $\det D^2 u = 1$ *on any cone* $\mathtt{C} \subset \mathbb{R}^2$ *are given by restrictions of a positive definite quadratic polynomial.*

Proof. Suppose that $u : \mathtt{C} \to \mathbb{R}$, $(x, y) \mapsto u(x, y)$ is such a solution, then without loss of generality, we can assume that the cone $\mathtt{C}$ lies in the upper half space $\{y \geq 0\}$. If we set $f(t) := \sqrt{u(t,1)}$, then

$$u(x,y) = u\left(y \cdot \frac{x}{y}, y \cdot 1\right) = y^2 u\left(\frac{x}{y}, 1\right) = y^2 f\left(\frac{x}{y}\right)^2.$$

We write $t = \frac{x}{y}$, we have

$$\begin{aligned} 1 &= \det D^2 u \\ &= \det \begin{pmatrix} 2f'(t)^2 + 2f(t)f''(t) & 2f(t)f'(t) - 2tf'(t)^2 - 2tf(t)f''(t) \\ 2f(t)f'(t) - 2tf'(t)^2 - 2tf(t)f''(t) & 2f(t)^2 - 4tf(t)f'(t) + 2t^2 f'(t)^2 + 2t^2 f(t)f''(t) \end{pmatrix} \\ &= 4f(t)^3 f''(t). \end{aligned}$$

So $f(t)$ must satisfy the ODE

$$f''(t) = \frac{c}{f(t)^3},$$

which can be integrated. First multiply both side by $f'(t)$, which gives

$$((f'(t))^2)' + c(f(t)^{-2})' = 0.$$

Integrating this, we have

$$f'(t)^2 + c(f(t))^{-2} = a \implies f'(t) = \sqrt{a - c(f(t))^{-2}} = \frac{\sqrt{af(t)^2 - c}}{f(t)},$$

for some constant c and a. One has

$$\frac{f\,\mathrm{d}f}{\sqrt{af^2 - c}} = \mathrm{d}t$$

Denote $u = af^2 - c$, one has $\frac{\mathrm{d}u}{2a\sqrt{u}} = \mathrm{d}t$, then $\frac{\sqrt{u}}{a} = t + K$, i.e.

$$f(t) = \sqrt{A(t - t_0)^2 + C}$$

for some constants A, t_0, C, which means

$$u(x, y) = y^2 f(\frac{x}{y})^2 = y^2[A(\frac{x}{y} - t_0)^2 + C] = A(x - t_0 y)^2 + Cy^2,$$

which is a positive definite quadratic polynomial. □

A corollary of this is that we can classify precisely the cones $(\mathtt{C}, \mathtt{C}') \subset (\mathbb{R}^2, \mathbb{R}^2)$ for which a homogenous optimal transport map of the form $((\mathtt{C}, dx), (\mathtt{C}', dy), u, v)$ exist.

Corollary 6.6. *Let $(\mathtt{C}, \mathtt{C}') \subset (\mathbb{R}^2_x, \mathbb{R}^2_y)$ be two convex cones, then a homogenous optimal transport map of the form $((\mathtt{C}, dx), (\mathtt{C}', dy), u, v)$ exists iff we are in one of the following four cases:*

1. *(Half-space case) Both $\mathtt{C}$ and $\mathtt{C}'$ are half-spaces and $l \cdot l' > 0$, where l and l' are the linear functions defining the half-space $\mathtt{C}$ and $\mathtt{C}'$ respectively. In this case, there is a* ***one-parameter*** *family of optimal maps between $\mathtt{C}$ and $\mathtt{C}'$.*
2. *(Acute/strongly oblique case) Both $\mathtt{C}$ and $\mathtt{C}'$ are strict cones and $\overline{\mathtt{C}'} \setminus \{0\} \subset int(\mathtt{C}^\circ)$. In this case, there is a* ***unique*** *homogeneous optimal transport map that maps $\mathtt{C}$ to $\mathtt{C}'$.*
3. *(Right angle case) Both $\mathtt{C}$ and $\mathtt{C}'$ are strict cones and $\mathtt{C}' = \mathtt{C}^\circ$, in which case there is a* ***one-parameter*** *family of homogeneous optimal transport maps from $\mathtt{C}$ to $\mathtt{C}'$.*
4. *(Obtuse case) Both $\mathtt{C}$ and $\mathtt{C}'$ are strict cones and $\overline{\mathtt{C}^\circ} \setminus \{0\} \subset int(\mathtt{C}')$. In this case, there is a* ***unique*** *homogeneous optimal transport map that maps $\mathtt{C}$ to $\mathtt{C}'$.*

Proof of Corollary. Assume such a homogeneous optimal transport map $((\mathtt{C}, dx), (\mathtt{C}', dy), u, v)$ exist, then Proposition 6.5 implies u and v are positive quadratic polynomials which are Legendre dual. Hence there exists an affine transformation A with $\det A = 1$ that puts u and v into its standard form, which means

$$A \cdot ((\mathtt{C}, dx), (\mathtt{C}', dy), u, v) = ((\tilde{\mathtt{C}}, dx), (\tilde{\mathtt{C}}', dy), \frac{|x|^2}{2}, \frac{|y|^2}{2}).$$

In this new form, the optimal transport map is just given by the identity map, hence we have $\tilde{\mathtt{C}} = \tilde{\mathtt{C}}'$, and the four cases correspond to different types of cones $\tilde{\mathtt{C}}$.

If $\tilde{\mathtt{C}}$ is a half-space (which, after a rotation can be assumed to be the upper half-space), that puts us into case 1. In this case, there is a 1-parameter family of optimal transport maps given by

$$u_\lambda(x) = \frac{\lambda x_1^2}{2} + \frac{x_2^2}{2\lambda}.$$

If $\tilde{\mathtt{C}}$ is an acute cone (i.e. cone angle $\theta < \frac{\pi}{2}$), then we are in case 2, and in this case the optimal transport map is unique.

If $\tilde{\mathtt{C}}$ has cone angle $\theta = \frac{\pi}{2}$, then we are in case 3. In this case after rotation it can be assumed that $\tilde{\mathtt{C}} = \{x_1 > 0, x_2 > 0\}$) and again there is a 1-parameter family of optimal transport maps given by $u_\lambda(x) = \frac{\lambda x_1^2}{2} + \frac{x_2^2}{2\lambda}$.

Finally if $\tilde{\mathtt{C}}$ is an obtuse cone, (i.e. cone angle $\theta > \frac{\pi}{2}$) then we are in case 4, and in this case, the homogeneous optimal transport map is unique as well. □

The next theorem says that if we are in case 4, the sections are always round.

Theorem 6.7. *Let $((\Omega, dx), (\Omega', dy), u, v)$ be an optimal transport map between convex domains in $\mathbb{R}^2$ satisfying $0 \in \partial\Omega \cap \partial\Omega'$ and $0 = u(0) = |\nabla u|(0)$. Let $(\mathtt{C}, \mathtt{C}')$ be the tangent cones of (Ω, Ω') at 0. If $(\mathtt{C}, \mathtt{C}')$ are both strict cones and $\overline{\mathtt{C}^\circ} \setminus \{0\} \subset int(\mathtt{C}')$, then the sections of (u, v) are round at 0.*

Proof. First we normalize $(\mathtt{C}, \mathtt{C}')$. By applying an affine transformation A with $\det A = 1$, we can arrange that

$$\mathtt{C} = \{x \geq 0, y \geq 0\}$$

and

$$\mathtt{C}' = \{y \geq -ax, x \geq -ay\}$$

for some $0 < a < 1$. Moreover since (Ω, Ω') is asymptotic to $(\mathtt{C}, \mathtt{C}')$ near zero, that means we can write

$$\Omega = \{(x, y) : y \geq f_+(x), x \geq f_-(y)\}$$

and

$$\Omega' = \{(x, y) : y \geq -ax + g_+(x), x \geq -ay + g_-(y)\}$$

where $f_\pm, g_\pm \geq 0$ are convex functions with $\lim_{t\to 0} \frac{f_\pm(t)}{t} = \lim_{t\to 0} \frac{g_\pm(t)}{t} = 0$.

From here, the basic idea is to show that if the sections are not round, then the direction of the long axis must align with the boundary of the cone $\mathtt{C}$. By symmetry, this must be true for both u and v, but that is inconsistent with the condition $\overline{\mathtt{C}^\circ} \setminus \{0\} \subset \mathrm{int}(\mathtt{C}')$.

Let E_h be the John Ellipsoid of $S_h(u, 0)$, which by uniform density (Lemma 4.2) is comparable to $S_h^c(u, 0)$. Suppose for contradiction that E_h is highly eccentric for $h \ll 1$. If we let $v_1(h), v_2(h)$ denote unit vectors in the directions of the long and short axis of E_h respectively, and $l_1(h), l_2(h)$ be the lengths of the long and short axis, then this means we have $v_1 \perp v_2$, and $l_1 \gg l_2$ for $h \ll 1$.

Now we claim that for any $\theta > 0$, there exists $h_0 > 0$ such that for all $0 < h < h_0$, the axis vectors must lie θ close to the two coordinate axis, that is $v_1(h), v_2(h) \in B_\theta(e_1) \cup B_\theta(e_2)$ for h sufficiently small. To see this we first note that since $l_1(h) \gg l_2(h)$ for h sufficiently small, therefore $\partial S_h(u, 0) \cap \partial\Omega$ is two points, which we call $(x_\pm, y_\pm)$ satisfying $f_+(x_+) = y_+$ and $f_-(y_-) = x_-$. We denote

$$T_h := \text{Convex Hull}(\{(0, 0), (x_+, y_+), (x_-, y_-)\})$$

and

$$R_h := \{x \geq 0, y \geq 0, y \geq a^{-1}(x - x_+), x \geq a^{-1}(y - y_-)\},$$

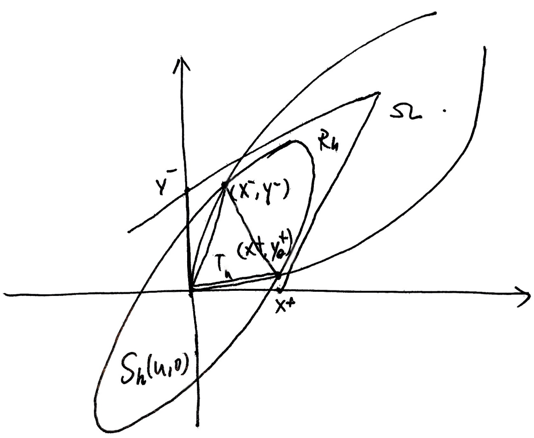
then we have

$$T_h \subset S_h(u, 0) \subset R_h.$$

Moreover without loss of generality, we can assume $x_+ \geq y_-$ (otherwise we flip the two coordinate axis), then there exist $c > 0$, and $x' \in T_h$ such that $B_{c|y_-|}(x') \subset T_h$, and there exist $C > 1$ such that $R_h \subset B_{C|x_+|}(0)$. Without lost of generality, we can assume that $x_+ \geq y_-$ (if not, then we can flip the two axis), then for any $C > 0$, there is sufficiently small h for which $x_+ \geq Cy_-$. If not, then $x_+ \sim y_-$ and

$$B_{c|x_+|}(x') \subset T_h \subset S_h(u, 0) \subset R_h \subset B_{C|x_+|}(0),$$

which would imply that $S_h(u, 0)$ are round.



Next we claim for any $\delta > 0$, there exist sufficiently small h for which $S_h(u,0) \subset P_{\delta|x_+|}$, where

$$P_{\delta|x_+|} = R_h \cap \{y \leq \delta|x_+|\}.$$

If not, then there exist $(x_0, \delta|x_+|) \in S_h$ with $x_0 \leq 2|x_+|$, which means

$$\text{ConvexHull}\{(0,0),(x_+,y_+),(x_0,\delta|x_+|)\} \subset S_h(u,0),$$

and if x_+ is sufficiently small, we have $y_+ < \frac{\delta}{2}x_+$, therefore

$$B_{c\delta|x_+|}(x') \subset \text{ConvexHull}\{(0,0),(x_+,y_+),(x_0,\delta|x_+|)\} \subset S_h(u,0)$$

which would mean that the sections $S_h(u,0)$ are round.

Thus for h sufficiently small

$$T_h \subset S_h(u,0) \subset P_{\delta|x_+|} \subset E(C|x_+|,\delta|x_+|).$$

If we pick δ sufficiently small, then it follows from this that the long axis of the John ellipsoid of $S_h(u,0)$ must point θ-close to the e_1-direction, which means $v_1 \in B_\theta(e_1)$.

Now we show this is a contradiction. The same argument applied to both u and v shows that we can find sequence $h_i \to 0$ for which $S_{h_i}(u,0)$ and $S_{h_i}(v,0)$ are becoming more and more eccentric, and the long axis of $S_{h_i}(u,0)$ point towards on the coordinate axis, while the long axis of $S_{h_i}(v,0)$ point towards an axis of $\mathtt{C}'$, moreover since we know by Lemma 4.2 that $|S_h(u,0)| \sim |S_h(v,0)| \sim h$, this implies there exist $x_h \in S_h(u,0)$ pointing in the direction of the long axis (i.e. $\frac{x_h}{|x_h|} \in B_\theta(e_i)$) with $|x_h| \gg h^{\frac{1}{2}}$, and there exist $y_h \in S_h(v,0)$ pointing in the direction of the long axis (i.e. $\frac{y_h}{|y_h|} \in B_\theta((\frac{1}{\sqrt{1+a^2}}, -\frac{a}{\sqrt{1+a^2}})))$ such that $|y_h| \gg h^{\frac{1}{2}}$. Since the axis of $\mathtt{C}$ and $\mathtt{C}'$ are not perpendicular, it follows that for h sufficiently small

$$|x_h \cdot y_h| \geq \delta|x_h||y_h| \gg h,$$

but this contradicts Proposition 2.3. $\square$

As a consequence of this, we can give a complete answer to Question 6.3 for polygonal domains in the plane.

Theorem 6.8. *Let $((\Omega,dx),(\Omega',dy),u,v)$ be an optimal transport map between two polygonal domains in $\mathbb{R}^2$. Suppose $0 \in \partial\Omega \cap \partial\Omega'$ with $0 = u(0) = |\nabla u|(0)$ and let $(\mathtt{C},\mathtt{C}')$ be the tangent cones of (Ω,Ω') at 0. Then the sections at 0 are round iff there is a homogeneous optimal transport map between $(\mathtt{C},\mathtt{C}')$. (i.e. iff we are in one of the situations of Corollary 6.6.)*

Proof. If there is no homogeneous optimal transport map between the tangent cones, by Theorem 6.2, the sections cannot be round at 0. On the other hand, if there is a homogeneous optimal transport map between the tangent cones $(\mathtt{C},\mathtt{C}')$, then we are in one of the four situations of Corollary 6.6, so we consider the 4 cases separately.

In the first case, we can reflect the optimal transport map across the boundary of the half-space, which turns the origin into an interior point, which means the sections are round.

In the second case, Theorem 6.4 tells us that the sections are always round.

In the third case, we can reflect the optimal transport map across one of the boundary sides to return to case 1.

In the fourth case, Theorem 6.7 tells us that the sections are always round. $\square$

Remark 6.9. *In the case when the domains (Ω,Ω') are not polygonal, if we are in case 2 and 4 of Corollary 6.6, then Theorem 6.4 and Theorem 6.7 still applies. The only cases where roundness is not known is cases 1 and 3. In those cases, any blow-up must live on the tangent cone, but there is a one-parameter family of homogeneous optimal transport maps between the tangent cones, and it is conceivable that there could be a solution that interpolates between different members of this family at different scales.*

6.3. **Homogeneous densities.** In this section, we focus on the situation when the domain $\Omega \subset \{x_n \geq 0\}$ is locally a strict cone, and the target is locally equal to the half-space $\{y_n \geq 0\}$ equipped with a homogenous density $g'(y) = y_n^k$. This situation arises in the study of complete Calabi-Yau metrics of Tian-Yau type in our previous work with S.-T. Yau [CTY], and as a limiting description of intermediate complex structure degeneration of Calabi-Yau manifolds, as shown in the work of Li [**?**]. In fact, together with S.-T. Yau, we have constructed homogeneous optimal transport maps between the cones of this type [CTY, Corollary 1.1]. In this situation, we show that the roundness of sections always hold.

Theorem 6.10. *Suppose that near the origin, Ω is equal to a strict cone $\mathtt{C} = \{x_n \geq \phi_{\mathtt{C}}(x')\}$, so $\Omega \cap B_{r_0} = \mathtt{C} \cap B_{r_0}$ and Ω' is locally a halfspace $\Omega' \cap B_{r_0} = \{y_n \geq 0\} \cap B_{r_0}$, and moreover assume the density $g(x)$ which is homogeneous of degree l and $g'(y) = y_n^k$. If in addition $k > l \geq 0$, then the sections of u are round. In particular, there exist $C > 1$, and $h_0 > 0$ such that for all $0 < h < h_0$, we have*

$$B_{C^{-1}h^{\frac{1}{1+\alpha}}} \subset S_h^c(u,0) \subset B_{Ch^{\frac{1}{1+\alpha}}}$$

and

$$B_{C^{-1}h^{\frac{1}{1+\frac{1}{\alpha}}}} \subset S_h^c(v,0) \subset B_{Ch^{\frac{1}{1+\frac{1}{\alpha}}}},$$

where $\alpha = \frac{n+l}{n+k} \in (0,1]$.

Proof. Let us E_h be the John ellipsoid of the section $S_h^c(v,0)$, then $E_h \cap \{y_n = 0\}$ is also an ellipsoid centered at 0 in $\mathbb{R}^{n-1}$, let $l_1(h) \leq \cdots \leq l_{n-1}(h)$ be the lengths of the principle axis of this ellipsoid, which after a rotation by $A \in SO(n-1)$, we can assume is in the direction of the coordinate axis. We also denote $d(h) := \sup\{x_n : x \in S_h^c(v,0)\}$ to be the height of E_h, then by John's lemma there exist $C > 1$ such that

$$C^{-1}E(l_1(h), \ldots, l_{n-1}(h), d(h)) \subset S_h^c(v,0) \subset CE(l_1(h), \ldots, l_{n-1}(h), d(h)).$$

By Proposition 2.3(2)(4), we also have

$$C^{-1}E\left(\frac{h}{l_1(h)}, \ldots, \frac{h}{l_{n-1}(h)}, \frac{h}{d(h)}\right) \subset S_h^c(u,0) \subset CE\left(\frac{h}{l_1(h)}, \ldots, \frac{h}{l_{n-1}(h)}, \frac{h}{d(h)}\right)$$

for $C > 1$ depending additionally on the doubling constant of g.

Since Ω' is locally a half-space, it follows that the centered sections of v has a uniform lower density bound

$$\frac{|S_h^c(v,0) \cap \Omega'|}{|S_h^c(v,0)|} \geq c > 0$$

and by Proposition 2.3, the centered sections of u also has a similar density bound

$$\frac{|S_h^c(v,0) \cap \Omega'|}{|S_h^c(v,0)|} \geq c > 0.$$

Moreover, since $\mathtt{C}$ is a strict cone, we have

$$\mathtt{C} \subset \mathtt{C}(B_R(0)) := \{(x', x_n) \in \mathbb{R}^n : x_n \geq \frac{1}{R}|x'|\}$$

for some $R > 1$. This implies the supporting hyperplanes of the sections $S_h(v,0)$ at points $y \in S_h(v,0) \cap \{y_n = 0\}$ have slope at most R, which implies

$$d(h) \leq CRl_i(h) \text{ for } i = 1, \ldots, n-1.$$

Moreover, by Proposition 2.3,

$$\nu(S_h(v,0)) \sim \int_{E(l_1(h),\cdots,l_{n-1}(h),d(h)) \cap \{y_n \geq 0\}} y_n^k dy \sim l_1(h) \cdots l_{n-1}(h) d(h)^{k+1}$$

and

$$\mu(S_h(u,0)) \sim \int_{E(\frac{h}{l_1(h)},\cdots,\frac{h}{l_{n-1}(h)},\frac{h}{d(h)})\cap \mathtt{C}} g(x)dx \sim \frac{h}{l_1(h)}\cdots\frac{h}{l_{n-1}(h)}(\frac{h}{d(h)})^{l+1}.$$

Therefore, the equation tells us that

$$l_1(h)\cdots l_{n-1}(h)d(h)^{k+1} \sim \frac{h}{l_1(h)}\cdots\frac{h}{l_{n-1}(h)}(\frac{h}{d(h)})^{l+1} \implies (l_1(h)\cdots l_{n-1}(h))^2 d(h)^{k+l+2} \sim h^{n+l}.$$

This implies that

$$d(h)^{2n+k+l} \lesssim h^{n+l} \implies d(h) \lesssim h^{\frac{n+l}{2n+k+l}}.$$

But the monotonicity formula tell us that

$$\mu(S_h(v,0)) \sim l_1(h)\cdots l_{n-1}(h)d(h)^{k+1} \gtrsim h^{\frac{n+l}{1+\frac{n+l}{n+k}}} \implies d(h)^{\frac{k-l}{2}} \gtrsim h^{\frac{(n+l)(k-l)}{2(2n+k+l)}}$$

since $k \geq l \geq 0$, this implies

$$d(h) \gtrsim h^{\frac{n+l}{2n+k+l}}.$$

Therefore we have $d(h) \sim l_i(h) \sim h^{\frac{n+l}{2n+k+l}}$, and the proposition is proved. ☐

Corollary 6.11. *Under the assumptions of Theorem 6.10, blow-ups exists and are homogeneous optimal transport maps between the corresponding tangent cones.*

Proof. This follows immediately from Theorem 4.6. ☐

Remark 6.12. *A consequence of this is that homogeneous solutions of the optimal transport problem constructed in* [CTY]*, which is expected to serve as a model at infinity for "generalized Tian-Yau metrics", arises as a blow-up limit for the optimal transport problem describing intermediate complex structure degenerations of Calabi-Yau hypersurfaces* [**?**]*. This provides further evidence that generalized Tian-Yau metrics should exist and should arise as bubbles in the intermediate complex structure degeneration of Calabi-Yau hypersurfaces.*

7. Flat directions and mixed homogeneity

Let $\mathtt{C} = \{x_n \geq \phi_{\mathtt{C}}(x')\} \subset \mathbb{R}^m$ be a strict cone. In this section, we consider optimal transport maps from $(\mathbb{R}^{n-m} \times \mathtt{C}, dx)$ to $(\mathbb{R}^{n-1} \times \mathbb{R}_+, y_n^k dy)$. This situation arises in our previous work with S.-T. Yau [CTY]. We'll see that in this case, the existence of flat-sides in the domain breaks the homogeneity, and the solutions exhibit mixed homogeneous behaviour.

7.1. **A Pogorelov estimate along flat directions.** We first prove Pogorelov-type estimates for solutions in the flat directions. When $\mathtt{C} = \mathbb{R}_+$ is a one-dimensional cone, a Pogorelov-type estimate was established by Jhaveri-Savin [JS, Proposition 4.1] using a doubling trick. The following estimate generalizes this result to arbitrary cones and general densities.

Proposition 7.1. *Let $\Omega \subset \mathbb{R}^n_x$ and $\Omega' \subset \mathbb{R}^n_y$. Suppose that the following structural conditions hold:*

(i) There is a smooth convex function $\phi_\Omega(x) = \phi_\Omega(x_2,\dots,x_{n-1})$ such that $\phi_\Omega(0) = 0$ and

$$\Omega \cap B_1(0) = \{x \in B_1(0) : x_n \geq \phi_\Omega(x_2,\dots x_{n-1})\}$$

(ii) There is a smooth convex function $\phi_{\Omega'}(y) = \phi_{\Omega'}(y_1,\dots,y_{n-1})$ such that $\phi_{\Omega'}(0) = 0$ and

$$\Omega' \cap B_1(0) = \{y \in B_1(0) : y_n \geq \phi_{\Omega'}(y_1,\dots y_{n-1})\}$$

(iii) There is a smooth, positive function $g(x)$ defined on $\overline{\Omega \cap B_1(0)}$ and such that $g(x) = g(x_2,\dots,x_n)$ is independent of x_1.

(iv) There is a smooth, positive function $g'(y)$ defined on $\overline{\Omega' \cap B_1(0)}$ and such that $g'(y)$ is log-concave and satisfies

$$\sum_{p=1}^{n} \frac{\partial g'}{\partial y_p} y_p \leq \kappa g'(y)$$

for some $\kappa > 0$ and for all $y \in \overline{\Omega' \cap B_1(0)}$.

Suppose that u is a convex function satisfying $0 = u(0) = |\nabla u|(0)$ and

$$\begin{aligned} g'(\nabla u) \det D^2 u = g(x), & \qquad \text{in } \Omega \cap B_1(0) \\ \frac{\partial u}{\partial x_n} \geq \phi_{\Omega'}(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_{n-1}}), & \qquad \text{in } \Omega \cap B_1(0) \\ \frac{\partial u}{\partial x_n} = \phi_{\Omega'}(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_{n-1}}), & \qquad \text{on } \partial\Omega \cap B_1(0) \end{aligned}$$

Let ℓ be the tangent plane to u at the origin, $h > 0$ and suppose that

$$S_h(u, 0) = \{u < \ell + h\} \Subset B_1(0) \cap \Omega.$$

Then we have a bound

$$u_{11}|u - \ell - h| \leq C(n, \kappa, \|u_1 - \ell_1\|_{L^\infty(S_h(0))}).$$

Proof. By subtracting a linear function from u, we may assume that $u(0) = |\nabla u(0)| = 0$. Let $S_h = S_h(u, 0)$ be the a section of height h centered at 0, and replace u with $u - h$. The argument is a Pogorelov type argument, making essential use of the boundary data. Consider the quantity

$$\Phi = \log u_{11} + \log |u| + \frac{1}{2} u_1^2$$

Let $x_* \in S_h$ be the point where Φ achieves its maximum. Either x_* is an interior point, or x_* lies on $\partial\Omega \cap S_h \cap B_1$.

Suppose first that x_* is an interior point. We may perform a shearing transformation preserving the x_1 direction, followed by a rotation in the $x_2, \dots, x_n$ directions in order to diagonalize $D^2 u(x_*)$. We note that since the x_1-direction is preserved, the structural conditions $(i) - (iv)$ are preserved under this affine change of coordinates. Differentiating the equation $g'(\nabla u) \det D^2 u = g(x)$ in the x_1-direction,

$$g u^{ij} u_{ij1} + \frac{\partial g'}{\partial y_p} u_{p1} \frac{g}{g'} = g_1 = 0. \quad \text{Since } g(x) = g(x_2, \cdots, x_n).$$

Differentiating again, we obtain

$$0 = \partial_1 \left(u^{ij} u_{ij1} + \frac{\partial}{\partial y_p} (\log g') u_{p1} \right) = -u^{ik} u^{jl} u_{kl1} u_{ij1} + u^{ij} u_{ij11} + \frac{\partial^2 \log g'}{\partial y_p \partial y_q} u_{p1} u_{q1} + \frac{1}{g'} \frac{\partial g'}{\partial y_p} u_{p11}.$$

Using that $D^2 u$ is diagonal at x_* (which yields $u^{pp} = \frac{1}{u_{pp}}$) we compute

$$\sum_p \frac{u_{1pp}}{u_{pp}} + \frac{1}{g'} \frac{\partial g'}{\partial y_1} u_{11} = 0 \tag{7.1}$$

$$\sum_p \frac{u_{11pp}}{u_{pp}} + \frac{1}{g'} \frac{\partial g'}{\partial y_p} u_{11p} = \sum_{p,q} \frac{u_{1pq}^2}{u_{pp} u_{qq}} - u_{11}^2 \frac{\partial^2 \log g'}{\partial y_1^2} \tag{7.2}$$

Differentiating Φ at x_* we have

$$\Phi_p = \frac{u_{11p}}{u_{11}} + \frac{u_p}{u} + u_{1p} u_1 = 0 \tag{7.3}$$

$$\Phi_{pp} = \frac{1}{u_{11}} u_{11pp} - \frac{(u_{11p})^2}{u_{11}^2} + \frac{u_{pp}}{u} - \frac{u_p^2}{u^2} + u_{1pp}u_1 + u_{1p}^2 \le 0 \tag{7.4}$$

Note that $u_{1p} = 0$ for all $p \neq 1$. Multiplying (7.4) by $u^{pp} = \frac{1}{u_{pp}}$ (Since D^2u is diagonal at x_*) and summing over p yields

(7.5)

$$\begin{aligned}
0 &\ge \sum_{1\le p\le n} \frac{u_{11pp}}{u_{11}u_{pp}} - \sum_{p=1}^n \frac{(u_{11p})^2}{u_{11}^2 u_{pp}} + \frac{n}{u} - \sum_{p=1}^n \frac{u_p^2}{u^2 u_{pp}} + u_1 \sum_{p=1}^n \frac{u_{1pp}}{u_{pp}} + u_{11} \\
&\ge \sum_{1\le p\le n} \frac{u_{11pp}}{u_{11}u_{pp}} - \sum_{p=1}^n \frac{(u_{11p})^2}{u_{11}^2 u_{pp}} + \frac{n}{u} - \sum_{p=2}^n \frac{u_{11p}^2}{u_{11}^2 u_{pp}} - \frac{u_1^2}{u^2 u_{11}} + u_1 \sum_{p=1}^n \frac{u_{1pp}}{u_{pp}} + u_{11} \quad \text{using (7.3)} \\
&= \left[\sum_{1\le p,q\le n} \frac{u_{1pq}^2}{u_{pp}u_{qq}u_{11}} - u_{11}\frac{\partial^2 \log g'}{\partial y_1^2} - \sum_{p=1}^n \frac{1}{g' u_{11}} \frac{\partial g'}{\partial y_p} u_{11p} \right] \\
&\qquad - \sum_{p=1}^n \frac{(u_{11p})^2}{u_{11}^2 u_{pp}} + \frac{n}{u} - \sum_{p=2}^n \frac{u_{11p}^2}{u_{11}^2 u_{pp}} - \frac{u_1^2}{u^2 u_{11}} + u_1 \sum_{p=1}^n \frac{u_{1pp}}{u_{pp}} + u_{11} \qquad \text{using (7.2)} \\
&\ge \sum_{2\le p,q\le n} \frac{u_{1pq}^2}{u_{pp}u_{qq}u_{11}} - u_{11}\frac{\partial^2 \log g'}{\partial y_1^2} - \sum_{p=1}^n \frac{1}{g' u_{11}} \frac{\partial g'}{\partial y_p} u_{11p} + \frac{n}{u} - \frac{u_1^2}{u^2 u_{11}} + u_1 \sum_{p=1}^n \frac{u_{1pp}}{u_{pp}} + u_{11} \\
&= \sum_{2\le p,q\le n} \frac{u_{1pq}^2}{u_{pp}u_{qq}u_{11}} - u_{11}\frac{\partial^2 \log g'}{\partial y_1^2} + \frac{1}{g'}\frac{\partial g'}{\partial y_1} u_{11}u_1 + \sum_{p=1}^n \frac{1}{g'}\frac{\partial g'}{\partial y_p}\frac{u_p}{u} + \frac{n}{u} - \frac{u_1^2}{u^2 u_{11}} + u_1 \sum_{p=1}^n \frac{u_{1pp}}{u_{pp}} + u_{11} \text{ using (7.3)}
\end{aligned}$$

By (7.1) we have

$$u_1 \sum_p \frac{u_{1pp}}{u_{pp}} + \frac{1}{g'}\frac{\partial g'}{\partial y_1} u_{11}u_1 = 0$$

Substituting this expression into (7.5) yields

$$0 \ge \sum_{2\le p,q\le n} \frac{u_{1pq}^2}{u_{pp}u_{qq}u_{11}} - u_{11}\frac{\partial^2 \log g'}{\partial y_1^2} + \frac{n}{u} - \frac{u_1^2}{u^2 u_{11}} + u_{11} + \sum_{p=1}^n \frac{1}{g'}\frac{\partial g'}{\partial y_p}\frac{u_p}{u}. \tag{7.6}$$

Since $u < 0$ in S_h, the structural property (iv): $g'(y)$ is log-concave, $\frac{\partial g'}{\partial y_p} y_p \le \kappa g'(y)$, $(y = \nabla u)$ gives

$$\frac{\partial^2 \log g'}{\partial y_1^2} \le 0 \quad \text{and} \quad \frac{1}{g'}\sum_p \frac{\partial g'}{\partial y_p}\frac{u_p}{u} \ge \frac{\kappa}{u}.$$

Thus, in total, at x_* we have

$$0 \ge \frac{n+\kappa}{u} - \frac{u_1^2}{u^2 u_{11}} + u_{11}.$$

Multiplying through the $|u|^2 u_{11}$ yields

$$|u|^2 u_{11}^2 + (n+\kappa)|u|u_{11} \le u_1^2 \;\Rightarrow |u|u_{11} \le C(n, \kappa, \|u_1\|_{L^\infty(S_h(0))}).$$

We now address the case when $x_* \in \partial\Omega \cap S_h \cap B_1(0)$. First we remark that since g, g' are smooth bounded strictly away from zero, the result of Chen-Liu-Wang [CLW], and the regularity theory of Caffarelli [C2, C3], u is smooth up to the boundary

By obliqueness, we can apply a shearing transformation in the $(x_2, \dots, x_n)$ directions which fixes the x_n direction, we may assume that e_n is the inward pointing normal vector to Ω at x_*. In other words, we have $\frac{\partial \phi_\Omega}{\partial x_j}(x_*) = 0$ for all $1 \le j \le n-1$. Similarly, by shearing transformation in the $(y_1, \dots, y_n)$ variables which fixes the y_n direction we may assume that e_n is the inward pointing normal vector to Ω' at $y_* = \nabla u(x_*)$. In other words, $\frac{\partial \phi_{\Omega'}}{\partial y_j}(y_*) = 0$ for all $1 \le j \le n-1$.

We may write the restriction of Φ to $\partial\Omega \cap S_h \cap B_1(0)$ as

$$\Phi\big|_{\partial\Omega}(x_1, x_2, \dots, x_{n-1}) = \Phi(x_1, \dots, x_{n-1}, \phi_\Omega(x_2, \dots, x_{n-1}))$$

Since Φ is maximized at x_* we have

$$0 = \frac{\partial \Phi}{\partial x_j} + \frac{\partial \Phi}{\partial x_n}\frac{\partial \phi_\Omega}{\partial x_j} = \frac{\partial \Phi}{\partial x_j} \quad \text{for } 1 \le j \le n-1. \tag{7.7}$$

Differentiating a second time yields

$$\begin{aligned} 0 &\ge \frac{\partial^2 \Phi}{\partial x_j^2} + \frac{\partial^2 \Phi}{\partial x_n \partial x_j}\frac{\partial \phi_\Omega}{\partial x_j} + \left(\frac{\partial^2 \Phi}{\partial x_j \partial x_n} + \frac{\partial^2 \Phi}{\partial x_n^2}\frac{\partial \phi_\Omega}{\partial x_j}\right)\frac{\partial \phi_\Omega}{\partial x_j} + \frac{\partial \Phi}{\partial x_n}\frac{\partial^2 \phi_\Omega}{\partial x_j^2} \\ &= \frac{\partial^2 \Phi}{\partial x_j^2} + \frac{\partial \Phi}{\partial x_n}\frac{\partial^2 \phi_\Omega}{\partial x_j^2} + 2\frac{\partial^2 \Phi}{\partial x_n \partial x_j}\frac{\partial \phi_\Omega}{\partial x_j} + \frac{\partial^2 \Phi}{\partial x_n^2}\left(\frac{\partial \phi_\Omega}{\partial x_j}\right)^2 \\ &= \frac{\partial^2 \Phi}{\partial x_j^2} + \frac{\partial \Phi}{\partial x_n}\frac{\partial^2 \phi_\Omega}{\partial x_j^2}\bigg|_{x_*} \end{aligned} \tag{7.8}$$

We now consider the derivative of Φ in the x_n direction. We have

$$\Phi_n = \frac{u_{11n}}{u_{11}} + \frac{u_n}{u} + u_1 u_{1n}$$

On the other hand, from the boundary data we have

$$u_n = \phi_{\Omega'}(u_1, \dots, u_{n-1})$$

and so

$$u_{1n} = \sum_{p=1}^{n-1} \frac{\partial \phi_{\Omega'}}{\partial y_p} u_{1p} \qquad u_{1n}(x_*) = \sum_{p=1}^{n-1} \frac{\partial \phi_{\Omega'}}{\partial y_p}(y_*) u_{1p}(x_*) = 0.$$

Differentiating again yields

$$u_{11n} = \sum_{1 \le p,q \le n-1} \frac{\partial^2 \phi_{\Omega'}}{\partial y_p \partial y_q} u_{1p} u_{1q} + \sum_{p=1}^{n-1} \frac{\partial \phi_{\Omega'}}{\partial y_p} u_{11p}.$$

By (7.7) we have

$$\Phi_p = \frac{u_{11p}}{u_{11}} + \frac{u_p}{u} + u_{1p} u_1 = 0 \quad \text{for } 1 \le p \le n-1$$

and so

$$\frac{u_{11n}}{u_{11}} = \sum_{1 \le p,q \le n-1} \frac{\partial^2 \phi_{\Omega'}}{\partial y_p \partial y_q} \frac{u_{1p} u_{1q}}{u_{11}} - \sum_{p=1}^{n-1} \frac{\partial \phi_{\Omega'}}{\partial y_p}\left(\frac{u_p}{u} + u_{1p} u_1\right).$$

This gives

$$\begin{aligned} \Phi_n &= \sum_{1 \le p,q \le n-1} \frac{\partial^2 \phi_{\Omega'}}{\partial y_p \partial y_q} \frac{u_{1p} u_{1q}}{u_{11}} - \sum_{p=1}^{n-1} \frac{\partial \phi_{\Omega'}}{\partial y_p}\left(\frac{u_p}{u} + u_{1p} u_1\right) + \frac{\phi_{\Omega'}}{u} + u_1 \sum_{p=1}^{n-1} \frac{\partial \phi_{\Omega'}}{\partial y_p} u_{1p} \\ &= \sum_{1 \le p,q \le n-1} \frac{\partial^2 \phi_{\Omega'}}{\partial y_p \partial y_q} \frac{u_{1p} u_{1q}}{u_{11}} + \frac{1}{|u|}\left(\sum_{p=1}^{n-1} \frac{\partial \phi_{\Omega'}}{\partial y_p} u_p - \phi_{\Omega'}\right). \end{aligned}$$

By the convexity of ϕ_Ω, the first term is non-negative. By the convexity of $\phi_{\Omega'}$ and $\phi_{\Omega'}(0) = 0$, the second term is non-negative as well. Hence we conclude that

$$\Phi_n(x_*) \ge 0.$$

In particular, we must have

$$\Phi_n(x_*) = 0 \quad \text{and} \quad \Phi_{nn}(x_*) \le 0. \tag{7.9}$$

Substituting this into (7.8) we obtain

$$\frac{\partial \Phi}{\partial x_j}(x_*) = 0 \quad \text{and} \quad \frac{\partial^2 \Phi}{\partial x_j^2}(x_*) \le 0. \tag{7.10}$$

Finally, since $u_{1n}(x_*) = 0$, we may perform a shearing transformation preserving the x_1 and x_n directions so that $D^2u(x_*)$ is diagonal. By structural condition (i), $\partial\Omega'$ is invariant under translations in the x_1 direction and so equations (7.1), (7.2) hold up to the boundary. Furthermore, by (7.9) and (7.10) we conclude that (7.3) and (7.4) also hold at x_*. We may therefore treat x_* as if it were an interior point, and the previous argument leads to the desired estimate. □

Remark 7.2. *In Proposition 7.1, the non-degeneracy of the measures $g(x)dx$ and $g'(y)dy$, as well as the smoothness of the boundaries Ω, Ω' can typically be relaxed by an approximation argument; see for example* [JS, Proposition 4.1]*. We given an example of such an argument in the proof of Theorem 7.3 below.*

7.2. **Shape of sections in mixed homogeneity case.** We now prove the main result of this section, which gives the shape of sections when the domains has flat side. As we can see, the flat sides breaks the homogeneity, and the solution exhibits mixed homogeneous behaviour.

Theorem 7.3. *Suppose that near the origin Ω is locally equal to $\mathbb{R}^{n-m} \times \mathtt{C}$ and Ω' is locally equal to $\mathbb{R}^{n-1} \times \mathbb{R}_+$, and $g(x) = 1, g'(y) = y_n^k$. Then there exist $C > 1$ such that for $h \ll 1$, we have*

$$B^{n-m}_{C^{-1}h^{\frac12}} \times B^m_{C^{-1}h^{\frac{1}{1+\frac{m}{m+k}}}} \subset S^c_h(u,0) \subset B^{n-m}_{Ch^{\frac12}} \times B^m_{Ch^{\frac{1}{1+\frac{m}{m+k}}}},$$

and

$$B^{n-m}_{C^{-1}h^{\frac12}} \times B^m_{C^{-1}h^{\frac{1}{1+\frac{m+k}{m}}}} \subset S^c_h(v,0) \subset B^{n-m}_{Ch^{\frac12}} \times B^m_{Ch^{\frac{1}{1+\frac{m+k}{m}}}}.$$

Proof. Let $S^c_h(v,0) \sim E(l_1, l_2, \dots, l_{n-1}, d(h))$ and $S^c_h(u,0) \sim E(\frac{h}{l_1}, \frac{h}{l_2}, \dots, \frac{h}{l_{n-1}}, \frac{h}{d(h)})$ and assume without loss of generality that

$$l_1 \le l_2 \le \dots \le l_{n-1}.$$

First we claim that by the Pogorelov estimate (Proposition 7.1) we have

$$l_1 \sim l_{n-m} \sim h^{\frac12}. \tag{7.11}$$

We first prove that $\ell_{n-m} \lesssim h^{\frac12}$ using Proposition 7.1 and an approximation argument. By the duality of sections, and rotational symmetry in the $(x_1, \dots, x_{n-m})$ variables, it suffices to prove the bound

$$\sup_{S_{h_0}(u,0)} u_{11} \le C$$

Suppose that u satisfies the assumptions of the theorem, and let $v = u^*$ be the Legendre dual of u. Fix some h_0 and some $\delta > 0$ so that $S_{h_0}(u,0) \subset B_{1-2\delta}$. Choose a sequence $\mathtt{C}_j \subset \mathbb{R}^m$ of smooth convex sets approximating $\mathtt{C}$ in $B_1(0)$, and choose also a sequence of sets $\Upsilon_j \subset \{y \in \mathbb{R}^n : y_n \ge 0\}$ which are dilations of $\Upsilon := \nabla u((\mathbb{R}^k \times \mathtt{C}) \cap B_1(0))$, such that for j sufficiently large

- we have the containment

$$\Upsilon_\delta := \nabla u(((\mathbb{R}^{n-m} \times \mathtt{C}) \cap B_{1-\delta}(0)) \Subset \Upsilon_j.$$

- the mass balancing condition holds:

$$\int_{(\mathbb{R}^{n-m} \times \mathtt{C}_j) \cap B_1(0)} dx = \int_{\Upsilon_j} (y_n + \frac{1}{j})^k dy$$

Let v_j be convex function solving the optimal transport problem

$$\begin{aligned} \det D^2 v_j &= (y_n + j^{-1})^k, \quad \text{in } \Upsilon_j \\ \nabla v_j(\Upsilon_j) &= (\mathbb{R}^{n-m} \times \mathtt{C}_j) \cap B_1(0) \end{aligned} \tag{7.12}$$

By the regularity theory for optimal transport maps [C1, C2, CLW, JS] we have that, for any j, $v_j \in C^{2,\alpha}(\overline{\Upsilon}_\delta)$ and the v_j are uniformly bounded in $C^{1,\alpha}(\overline{\Upsilon}_\delta)$, and hence v_j converge to v in $C^{1,\alpha}(\overline{\Upsilon}_\delta)$. In particular, if u_j denotes the Legendre transform of v_j, then for j large u_j satisfies

$$\begin{aligned} (\frac{\partial u_j}{\partial x_n} + j^{-1})^k \det D^2 u = 1, &\quad \text{in } (\mathbb{R}^{n-m} \times \mathtt{C}_j) \cap S_{h_0}(u_j, 0) \\ \frac{\partial u_j}{\partial x_n} \geq 0, &\quad \text{in } (\mathbb{R}^{n-m} \times \mathtt{C}_j) \cap S_{h_0}(u_j, 0) \\ \frac{\partial u_j}{\partial x_n} = 0, &\quad \text{on } (\mathbb{R}^{n-m} \times \partial \mathtt{C}_j) \cap S_{h_0}(u_j, 0) \end{aligned} \tag{7.13}$$

We may therefore apply Proposition 7.1 uniformly to u_j with $\kappa = k$, and the result follows by taking the limit as $j \to \infty$.

Next we prove the reverse estimate $\ell_1 \gtrsim h^{\frac{1}{2}}$, which will follow from the estimate

$$\sup_{S_{h_0}(v,0)} v_{11} \leq C$$

for some C, h_0. Since the approximation argument is similar to the preceding argument, we only sketch the details. We choose a sequence of smooth convex functions $\phi_{\mathtt{C}_m}$ such that $\phi_{\mathtt{C}_m}(0) = 0$, and $\phi_{\mathtt{C}_m}$ converge to $\phi_{\mathtt{C}}$ uniformly on compact sets. Let

$$\mathtt{C}_m = \{x_n \geq \phi_{\mathtt{C}_m}(x_1, \ldots, x_{n-1})\}$$

We now solve a sequence of optimal transport problems with smooth non-degenerate measures given by dx and $(y_n + \varepsilon)^k dy$, and targets contained in $\mathtt{C}_m$. The result follows by applying the preceding Pogorelov interior estimate, and passing to the limit as $m \to \infty$.

We have thus established (7.11). By similar arguments to those in the proof of Theorem 6.10 we will show that

$$d(h) \lesssim l_{n-m+1} \leq \cdots \leq l_{n-1}.$$

Indeed, let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and let $S_h^{c,i}(v,0) = S_h^c(v,0) \cap \mathrm{Span}\{e_i, e_n\}$. Define

$$\begin{aligned} -\ell_i(h) &= \min\{x_i : (x_i, 0) \in S_h^{c,i}(v,0)\} \\ r_i(h) &= \max\{x_i : (x_i, 0) \in S_h^{c,i}(v,0)\} \end{aligned}$$

and note that $\ell_i(h) \sim r_i(h)$ by balancing. Since $\mathtt{C}$ is a strict cone, we have

$$\phi_{\mathtt{C}}(x_{n-m+1}, \ldots, x_{n-1}) \geq c \max_{n-m+1 \leq i \leq n-1} |x_i|$$

for some uniform $c > 0$ depending only on $\mathtt{C}$. For any $n - m + 1 \leq i \leq n - 1$, the boundary values yield

$$\begin{aligned} \frac{\partial v}{\partial y_n}(-\ell_i(h) e_i) &= \phi_{\mathtt{C}}(\frac{\partial v}{\partial y_{n-m+1}}(-\ell_i(h) e_i), \ldots, \frac{\partial v}{\partial y_{n-1}}(-\ell_i(h) e_i)) \\ &\geq c |\frac{\partial v}{\partial y_i}(-\ell_i(h) e_i)| \\ &\geq c \frac{\ell_i(h)}{h} \end{aligned}$$

where we used convexity of v in the last line. From now on we consider the two dimensional subspace $\mathbb{R}^2 = \mathrm{Span}\{e_i, e_n\} \subset \mathbb{R}^n$ for $n-m+1 \le i \le n$. Let

$$\vec{\eta} = \left(\frac{\partial v}{\partial y_i}(-\ell_i(h)e_i), \frac{\partial v}{\partial y_n}(-\ell_i(h)e_i)\right).$$

Then we have shown that $\vec{\eta}\cdot e_n > c|\vec{\eta}\cdot e_i|$, and the line

$$(y_i + \ell_i(h), y_n)\cdot \eta = 0$$

is a supporting line for $S_h^{c,i}(v,0)$,intersecting the line $\{r_i(h)e_i + te_n : t \in \mathbb{R}\}$ at a point $r_i(h)e_i + t_* e_n$ where $t_* \sim \ell_i(h)$. Therefore,

$$d_h \lesssim \min_{n-m+1\le i\le n-1}\{\ell_i(h), r_i(h)\} \lesssim l_{n-m+1}$$

as desired.

From the mass balancing condition imposed by the equation, we have

$$\frac{h^n}{l_1 \dots l_{n-1} d(h)} \sim |S_h(u,0)| \sim \int_{S_h(v,0)} y_n^k dy \sim l_1 \cdots l_{n-1} d(h)^{k+1}$$

which implies

$$h^m \sim (l_{n-m+1}\cdots l_{n-1})^2 d(h)^{k+2} \gtrsim d(h)^{k+2+2(m-1)}$$

and so

$$|S_h(u,0)| \lesssim h^{\frac{n-m}{2}+\frac{m}{1+\frac{m}{m+k}}}.$$

We claim that a comparable lower volume bound holds

Lemma 7.4. *In the above setting, there is a constant $c > 0$ so that*

$$|S_h(u,0)| \ge c h^{\frac{n-m}{2}+\frac{m}{1+\frac{m}{m+k}}} \tag{7.14}$$

for all $h \le 1$.

Assuming Lemma 7.4, the shape of the sections follows as in Theorem 6.10. □

It only remains to prove Lemma 7.4.

Proof of Lemma 7.4. We argue by induction on m. First note that it suffices to prove (7.14) for all h sufficiently small, up to redefining c. When $m = 1$, the result follows from the work of Jhaveri-Savin [JS, Theorem 1.4]. Suppose that (7.14) is not true. Set

$$\gamma = \frac{n-m}{2} + \frac{m}{1+\frac{m}{m+k}}$$

For $\varepsilon > 0$ define the set of "ε-bad scales" to be

$$\mathfrak{B}(\varepsilon) = \{h \in \mathbb{R}_+ : 0 < h \le 1 \text{ and } \varepsilon h^\gamma \le |S_h(u,0)| \le 2\varepsilon h^\gamma\}$$

and let

$$\widehat{\mathfrak{B}}(\varepsilon) := \bigcup_{\tau\le\varepsilon} \mathfrak{B}(\tau) = \{h \in \mathbb{R}_+ : 0 < h \le 1 \text{ and } |S_h(u,0)| \le 2\varepsilon h^\gamma\}$$

Suppose that, for some $\varepsilon > 0$ we have $\inf \widehat{\mathfrak{B}}(\varepsilon) = \delta > 0$. Then, for all $h < \delta$ we have

$$|S_h(u,0)| \ge 2\varepsilon h^\gamma$$

which is the desired result. Thus we may assume that there is a sequence $h_i \searrow 0$, and a sequence $\varepsilon_i \searrow 0$ such that $h_i \in \mathfrak{B}(\varepsilon_i)$. We will show that this scenario leads to a contradiction. The main technical tool is the following claim:

Claim 7.5. *Fix a constant $K \ge 1$. There exists h_0, ε_0, and $\lambda_0 \in (0,1)$ such that, if $h < h_0$, and $\varepsilon < \varepsilon_0$, and $\varepsilon h^\gamma \le |S_h(u,0)| \le 2\varepsilon h^\gamma$, then for some $h' \in (\lambda_0 h, h)$ we have $|S_{h'}(u,0)| \ge 2K\varepsilon(h')^\gamma$.*

We now prove (7.14) and hence Lemma 7.4, assuming Claim 7.5. First, observe that by convexity we have the containment

$$\frac{1}{2}S_h(u,0) \subset S_{\frac{h}{2}}(u,0)$$

and hence, if $h \in \mathfrak{B}(\varepsilon)$

$$|S_{\frac{h}{2}}(u,0)| \geq 2^{-n}|S_h| \geq 2^{-n+\gamma}\varepsilon\left(\frac{h}{2}\right)^{\gamma}$$

In particular, we have

$$h \in \mathfrak{B}(\varepsilon) \Rightarrow \frac{h}{2} \notin \widehat{\mathfrak{B}}(2^{-n+\gamma}\varepsilon). \tag{7.15}$$

We apply the claim with $K = 2^{n-\gamma} > 1$. Let $\varepsilon_0, h_0, \lambda_0$ be as in the claim. Let

$$h_1 = \inf\{h < h_0 : h \notin \widehat{\mathfrak{B}}(\varepsilon_0)\}.$$

By (7.15) we have $h_1 \in \mathfrak{B}(\varepsilon_1)$ for some $\varepsilon_1 \in (K^{-1}\varepsilon_0, \varepsilon_0)$. By Claim 7.5 we conclude that there is an $h_2 \in (\lambda_0 h_1, h_1)$ such that $h_2 \notin \widehat{\mathfrak{B}}(K\varepsilon_1)$. In particular, we obtain

$$h_2 \in \mathfrak{B}(\varepsilon_2) \quad \text{ for } \varepsilon_2 > K\varepsilon_1 > \varepsilon_0$$

Thus, for every $h \in (h_2, h_1) \subset (\lambda_0 h_1, h_1)$ we have

$$|S_h(u,0)| \geq |S_{h_2}(u,0)| \geq \varepsilon_0 h_2^{\gamma} \geq \varepsilon_0 \lambda_0^{\gamma} h_1^{\gamma} \geq \varepsilon_0 \lambda_0^{\gamma} h^{\gamma}$$

We can now repeat the argument, taking

$$h_3 = \inf\{h < h_2 : h \notin \widehat{\mathfrak{B}}(\varepsilon_0)\},$$

and proceeding as before. It follows that

$$|S_h(u,0)| \geq \varepsilon_0 \lambda_0^{\gamma} h^{\gamma} \quad \text{ for all } h < h_0$$

which contradicts our assumption regarding the existence of the sequence (h_i, ε_i). □

In order to complete the proof of Lemma 7.4 (and thereby Theorem 7.3), it only remains to prove Claim 7.5.

Proof of Claim 7.5. The proof is by contradiction. If the claim is not true, then there exist $h_p, \varepsilon_p, r_p, \delta_p, \mu_p \in (0,1)$ such that $h_p \searrow 0$, $\varepsilon_p \searrow 0$, $\mu_p \searrow 0$ as $p \to \infty$, and

$$r_p < h_p, \quad \delta_p < \varepsilon_p, \quad r_p \in \mathfrak{B}(\delta_p)$$

and such that, for all $h' \in (\mu_p r_p, r_p)$ we have $h' \in \widehat{\mathfrak{B}}(K\delta_p)$. By assumption we have $r_p \to 0$ and

$$\frac{|S_{r_p}(u,0)|}{r_p^{\gamma}} \leq 2\delta_p \to 0.$$

From the preceding analysis of the sections, this can only occur if $\frac{d(r_p)}{l_{n-1}(r_p)} \to 0$. Hence, the rescaled functions

$$u_p(x) := \frac{u(\frac{r_p}{l_1(r_p)}x_1, \ldots, \frac{r_p}{l_{n-1}(r_p)}x_{n-1}, \frac{r_p}{d(r_p)}x_n)}{r_p}$$

satisfy

$$\det D^2 u_p = \frac{r_p^n}{(l_1(r_p)\cdots l_{n-1}(r_p))^2 d(r_p)^{k+2}} \frac{1}{(u_p)_n^k}$$

and since $\frac{h^n}{(l_1 \cdots l_{n-1})^2 d(h)^{k+2}} \sim 1$, we can take a convergent subsequence converging to u_∞, which is a limiting optimal transport map $((\mathbb{R}^{n-m+j} \times \tilde{\mathtt{C}}, dx), (\mathbb{R}^{n-1} \times \mathbb{R}_+, y_n^k dy), u_\infty, v_\infty)$ for some $j > 0$ and some strict cone $\tilde{\mathtt{C}} \subset \mathbb{R}^{m-1-j} \times \mathbb{R}_+$. Therefore, by the induction hypothesis, there are constants c, C so that for all $h \leq 1$ there holds

$$ch^{\frac{n-m+p}{2}+\frac{m-p}{1+\frac{m-p}{m-p+k}}} \leq |S_h(u_\infty, 0)| \leq Ch^{\frac{n-m+p}{2}+\frac{m-p}{1+\frac{m-p}{m-p+k}}}.$$

If we denote $\beta := \frac{n-m+j}{2} + \frac{m-j}{1+\frac{m-j}{m-j+k}}$, then since $j > 0$, we have $\beta < \gamma$. Fix Λ large so that

$$\frac{1}{2}\frac{c}{C}\Lambda^{\gamma-\beta} > 4K.$$

Then, for p sufficiently large we have

$$\begin{aligned} |S_{\frac{r_p}{\Lambda}}(u, 0)| \geq \frac{c}{2C}\frac{1}{\Lambda^\beta}|S_{r_p}(u, 0)| &= \Lambda^{\gamma-\beta}\frac{c}{2C}\frac{1}{\Lambda^\gamma}|S_{r_p}(u, 0)| \\ &\geq 4K\frac{1}{\Lambda^\gamma}|S_{r_p}(u, 0)| \\ &\geq 4K\frac{1}{\Lambda^\gamma}\delta_p r_p^\gamma \end{aligned}$$

and so $\frac{r_p}{\Lambda} \notin \widehat{B}(K\delta_p)$ for p large, a contradiction. $\square$

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